*The Precise Determination of the Relative Orientation between Two Photographs Taken at a Single Exposure Station, and a Measure of Its Accuracy**

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ABSTRACT: *A precise solution to the problem of relative orientation between two photographs taken at a single exposure station is presented. This determination utilizes a redundancy of data, a rigorous least squares solution, and differential techniques requiring iterations. The method is expanded to include the determination of the absolute orientation of the second photograph, providing that the absolute orientation of the first photograph has been predetermined. The probable error of the orientation is achieved as a by-product of the solution.*

IN HIS recent paper, Arthur presents a development for the direct solution to the problem of relative orientation between two photographs taken from the same exposure station.[1] His presentation is an improvement over that offered by Faulds in an earlier paper.[2] Arthur indicates that the direct solution appears to be of academic interest only, but may be useful to determine a good first approximation in the case of twin-camera photography. If he is referring to 20 degree convergent photography, then a very good first approximation is already available in the nominal set-up of the convergent camera: 20 degrees y-tilt, zero x-tilt and zero swing.

Arthur also indicates that this method using two points contains a redundancy of information and does suggest that differential methods using redundant data and least squares techniques may be employed to achieve improved results. This suggestion is in line with the fact that although a direct solution can be obtained using the true least squares method, that the procedures are very complicated and highly impractical. Although the normal equations can be readily established, the solution can feasibly be accomplished only by linearization of the normal equations. It would therefore appear best to linearize the original functions and solve by successive approximations.

Such a method, using a rigorous least squares approach has been formulated at Air Force Missile Test Center, and will eventually be coded for automatic computation. This exact least squares method differs from the many least squares techniques for problems involving overdetermination in that the minimization is applied to residuals of the original observations, the plate measurements. It also leads to the most accurate determination of the relative orientation.

The transformation of direction cosines between the two rectangular reference systems associated with the two camera plates is given by the condition equations:

* Publication and distribution of this paper have been approved by the Administrative Contracting Officer, Air Force Missile Test Center. This paper extracted from RCA Data Reduction Technical Report No. 38, "The Precise Determination of the Orientation of the Second of Two Overlapping Photographs Taken at a Single Exposure Station," August, 1957, by G. H. Rosenfield.

$$
\begin{aligned}
l' &= Al + Bm + Cn \\
m' &= A'l + B'm + C'm \\
n' &= Dl + Em + Fn
\end{aligned}
\tag{1}
$$

where the orthogonal matrix representing the unknown relative orientation between the two photographs is

$$
M = \begin{bmatrix} A & B & C \\ A' & B' & C' \\ D & E & F \end{bmatrix} \tag{2}
$$

in which the elements of the matrix may be expressed in terms of the rotational elements of relative orientation, e.g., ϕ, ω, κ. The particular rotational elements to be used be arbitrarily selected and the elements of the matrix depend upon the definitions given to the selected rotational elements. In the following development, ϕ, ω, κ have been selected for use.

Since the direction cosines l, m, and n, and l', m', and n', used in Equation (1) are orthogonal in character, only two of the three equations are independent, and the third can be expressed in terms of the other two. As a result only two equations from each pass point are used.

The condition equations therefore take the form:

$$
\begin{aligned}
e &= l' - Al - Bm - Cn = 0 \\
e' &= m' - A'l - B'm - C'n = 0
\end{aligned}
\tag{3}
$$

These condition equations can be expressed in terms of the coordinate measurements:

$$
\begin{aligned}
e &= \frac{x'}{R'} - A\frac{x}{R} - B\frac{y}{R} - C\frac{f}{R} = 0 \\
e' &= \frac{y'}{R'} - A'\frac{x}{R} - B'\frac{y}{R} - C'\frac{f}{R} = 0
\end{aligned}
\tag{4}
$$

where

x, y = coordinate measurements on the first photograph

x', y' = coordinate measurements on the second photograph and f and f' = the focal lengths of the first and second photographs.

$$
R = \sqrt{x^2 + y^2 + f^2} \quad \text{and} \quad R' = \sqrt{x'^2 + y'^2 + f'^2}
$$

Since x, y, and x', y', are the true coordinate measurements which will satisfy these condition equations, corrections must be applied to the original observations to obtain the true coordinates; therefore

$$
\begin{aligned}
x &= x_0 + v_x \quad \text{and} \quad & x' &= x_0' + v_x' \\
y &= y_0 + v_y & y' &= y_0' + v_y'
\end{aligned}
\tag{5}
$$

where x_0, y_0 and x_0', y_0', denote the actual observations and the v's are the measuring residuals.

By the same reasoning the rotational elements of relative orientation may be expressed as

$$
\begin{aligned}
\phi &= \phi_0 + \delta\phi \\
\omega &= \omega_0 + \delta\omega \\
\kappa &= \kappa_0 + \delta\kappa
\end{aligned}
\tag{6}
$$

where $\phi_0\omega_0$ and K_0 are approximation values to the orientation parameters.

Substituting Equations (5) and (6) into (4) and linearizing the result by Taylor's series leads to expressions of the form:

$$
\begin{aligned}
e &= a_1 v_x + a_2 v_y + a_3 v_{x'} + a_4 v_{y'} + b_1\delta\phi + b_2\delta\omega + b_3\delta\kappa + \epsilon = 0 \\
e' &= a_1' v_x + a_2' v_y + a_3' v_{x'} + a_4' v_{y'} + b_1'\delta\phi + b_2'\delta\omega + b_3'\delta\kappa + \epsilon' = 0
\end{aligned}
\tag{7}
$$

where

$$
\begin{aligned}
\epsilon &= \frac{x_0'}{R_0'} - A_0\frac{x_0}{R_0} - B_0\frac{y_0}{R_0} - C_0\frac{f}{R_0} \\
\epsilon' &= \frac{y_0'}{R_0'} - A_0'\frac{x_0}{R_0} - B_0'\frac{y_0}{R_0} - C_0'\frac{f}{R_0}
\end{aligned}
\tag{8}
$$

in which A_0, B_0, C_0 etcs., denote the values of A, B, C, etc. resulting from the approximations ϕ_0, ω_0, κ_0 and where R_0, R_0' are the values of R, R' resulting from the actual measurements: x_0, y_0, x_0', y_0'. The coefficients $a_1, a_2 \cdots$, b_1, b_2, b_3, etc. are defined by

$$
\begin{array}{llll}
a_1 = \dfrac{\partial\epsilon}{\partial x_0} & a_1' = \dfrac{\partial\epsilon'}{\partial x_0} & b_1 = \dfrac{\partial\epsilon}{\partial\phi_0} & b_1' = \dfrac{\partial\epsilon'}{\partial\phi_0} \\
a_2 = \dfrac{\partial\epsilon}{\partial y_0} & a_2' = \dfrac{\partial\epsilon'}{\partial y_0} & b_2 = \dfrac{\partial\epsilon}{\partial\omega_0} & b_2' = \dfrac{\partial\epsilon'}{\partial\omega_0} \\
a_3 = \dfrac{\partial\epsilon}{\partial x_0'} & a_3' = \dfrac{\partial\epsilon'}{\partial x_0'} & b_3 = \dfrac{\partial\epsilon}{\partial\kappa_0} & b_3' = \dfrac{\partial\epsilon'}{\partial\kappa_0} \\
a_4 = \dfrac{\partial\epsilon}{\partial y_0'} & a_4' = \dfrac{\partial\epsilon'}{\partial y_0'} & &
\end{array}
\tag{9}
$$

Introducing the subscript "i" to refer to the i[th] pass point, the following matrices are defined:

$$
\begin{aligned}
A_i &= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1' & a_2' & a_3' & a_4' \end{bmatrix}_i \qquad
B_i = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_1' & b_2' & b_3' \end{bmatrix} \\
v_i &= \begin{bmatrix} v_x \\ v_y \\ v_x' \\ v_y' \end{bmatrix}_i \qquad
\delta = \begin{bmatrix} \delta\phi \\ \delta\omega \\ \delta K \end{bmatrix} \qquad
\epsilon_i = \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}_i
\end{aligned}
\tag{10}
$$

The pair of linearized condition equations for the i[th] pass point may therefore be written in matrix form as

$$
A_i v_i + B_i\delta_i + \epsilon_i = 0 \tag{11}
$$

Furthermore, the total collection of linearized condition equations for a ser of "n" pass points may be expressed in matrix form as

$$Av + B\delta + \epsilon = 0 \tag{12}$$

where

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & A_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ B_n \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \cdot \\ \cdot \\ \epsilon_n \end{bmatrix} \tag{13}$$

In general, it may be assumed that all observations are of equal weight. Then, according to Brown,[3] the normal equations for the least squares adjustment for condition equations of the generalized form (12) are given by

$$N\delta + c = 0 \tag{14}$$

and the formal solution to the normal equation is

$$\delta = -N^{-1}c \tag{15}$$

where

$$N = [B^T(AA^T)^{-1}B], \qquad c = [B^T(AA^T)^{-1}\epsilon] \tag{16}$$

The superscript "T" denotes transposition and the superscript "-1" denotes inversion.

Solution of the normal equations leads to values of $\delta\phi$, $\delta\omega$, $\delta\kappa$, for correction to the original approximation values. Successive iteration leads to the desired orientation of the second camera relative to the first.

The expansion of this solution from one of relative orientation to the more general case of absolute orientation is accomplished very simply by a series of matrix transpositions and multiplications.

Let M_1 = the absolute orientation matrix of the first photograph, and M = the computed orientation matrix for the two photographs. Then

$$M_2 = MM_1 \tag{17}$$

where M_2 = the computed absolute orientation matrix of the second photograph.

The above computations were performed on a desk calculator using a three pass point solution with first approximation values for the second photograph taken as the nominal convergent photograph. Fictitious convergent photography with 150 mm. focal length was used in the example. The first approximation values differed from the true orientation parameters by the amounts:

$$\delta\phi = 59'\ 59.''03, \qquad \delta\omega = 29'\ 59.''99, \qquad \delta\kappa = -\ 5'\ 01.''14.$$

As a result of the fifth iteration, the relative orientation was recaptured to an accuracy of:

$$\phi = +\ 0.''27, \qquad \omega = -\ 0.''95, \qquad \kappa = +\ 4.''59.$$

Since fictitious data with no introduced error were used, further iteration would have caused the residuals to converge to zero and the original orientation to be recaptured.

The mean error of the computed orientation parameters was obtained by multiplying the inverse of the normal equation coefficient matrix (N^{-1}) by the unit variance (an estimated reading error, since that is the only error in the fictitious photography). Assuming a reading error of 5 microns (the usual in aerial photogrammetry) the probable error of determining the orientation by this method is approximately: $\sigma_\phi = 5$ seconds of arc, $\sigma_\omega = 5$ seconds of arc, and $\sigma_K = 9$ seconds of arc.

Many approximate solutions using the method of least squares can be devised for this same problem. One which is immediately apparent minimizes directly the sum of the squares of the differences between the l' and m' values as known from the first photograph and computed from the approximate second photograph. This method is not theoretically correct; does not minimize that function wherein the independent random error lies (the original measurements); and does not lead to that important by-product of the true least squares method, the relative covariance matrix, from which the probable error of the solution is obtained.

References

1. Arthur D. W. G., "Relative Orientation of Photographs taken from the Same Station," PHOTOGRAMMETRIC ENGINEERING, Vol. XXII, No. 5, December 1956, pp. 935–938.
2. Faulds, A. H., "Determination of Relative Orientation for Two Overlapping Photographs Taken at a Common Exposure Station," PHOTOGRAMMETRIC ENGINEERING, Vol. XXII, No. 2, April 1956, pp. 392–397.
3. Brown, D. C., "A Treatment of Analytical Photogrammetry," RCA Data Reduction Technical Report No. 39, August 1957, Appendix A, "A Treatment of the General Problem of Least Squares and the Associated Error Propagation."

PHOTOGRAMMETRY COVERS BROAD FIELD

There is more to photogrammetry than mapping, however. The photography is not confined to an aerial exposure station for the purpose of topographic mapping. The metrical photography may be exposed from the air, ground, underground, water, underwater, or within the confines of a building for the purposes of topographic mapping, astronomy, ballistics, architure, medicine, anthropology, zoology, physics, biology, geology, meteorology, hydrography, deformations, criminalistics, industrial measurements, physical education, aviation, military intelligence, or determining the dimensions of an individual for a tailor-made suit. Metrical photography is capable of determining the position of points, the orientation and length of lines, the area of surfaces, and the volume of solids.

Metrical photography can also be used to determine time, and quantities integrated with time, such as velocity and acceleration.

The key to the capabilities of metrical photography is that a direct measurement in image space can be used to determine an indirect measurement in object space.

Excerpted from "Metrical Photography" by G. T. McNeil, PHOTOGRAMMETRIC ENGINEERING, *Vol. XXIII, no. 4, p. 670.*

"With cameras, computers, and electronic instruments they're mapping the world faster and more accurately, and finding for industry the wealth buried beneath its surface."

Excerpt from "Aerial Mappers Speed an Ancient Art," Business Week, *Oct. 19, 1947.*