A New Method for Analytical Radial Triangulation

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ABSTRACT: *In this paper the analytical radial triangulation is founded upon image coordinate measurements only. Simple formulae systems are derived for the computations. Methods earlier derived by the author for the corrections of systematic errors are applied in order to obtain image coordinates of highest possible precision. Since the errors of the fundamental measurements can be regarded as of mainly accidental character, strict adjustment procedures, according to the method of the least squares, can be correctly applied to the treatment of discrepancies in superfluous control points. The error propagation of the radial triangulation can be further studied in a theoretically correct way. The accuracy of the radial triangulation can be considerably increased in comparison with the results which have been obtained up to now.*

Under normal conditions only simple and common instruments are needed for the necessary measurements.

Investigations based upon these principles are being performed at the Division of Photogrammetry of the Institute of Technology in Stockholm.

THE FUNDAMENTAL FORMULAE

WE ASSUME image coordinates x', y' and x'', y'' to be measured in the two overlapping pictures in Figure 1. As a preparatory action we assume only accidental errors to be present in the measurements. The origins of the coordinate systems are located in the principal points and the x'- and x''-axes are directed along the base line.

The coordinates x, y of an intersected point can be analytically determined from the image coordinates and the base b in the following way (Figure 1),

We have:

$$x = r \cos \alpha \tag{1}$$

$$y = r \sin \alpha. \tag{2}$$

But from the sine-theorem we have:

$$r = \frac{b \sin \beta}{\sin (\alpha + \beta)} = \frac{b \sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta} = \frac{b}{\sin \alpha \cot \beta + \cos \alpha}. \tag{3}$$

From Figure 1 we find:

$$\cos \alpha = \frac{x'}{r'} \tag{4}$$

$$\sin \alpha = \frac{y'}{r'} \tag{5}$$

$$\cot \beta = -\frac{x''}{y''}. \tag{6}$$

Substituting (4)–(6) into (3) and then (3)–(5) into (1) and (2) we find after simple rearrangements:

$$x = \frac{b}{1 - \dfrac{y'x''}{x'y''}} \tag{7}$$

$$y = \frac{y'}{x'} \cdot \frac{b}{1 - \dfrac{y'x''}{x'y''}} \tag{8}$$

or

$$y = \frac{b}{\dfrac{x'}{y'} - \dfrac{x''}{y''}}.$$

From the simple expressions (7) and (8) the image coordinates can easily be transformed into the coordinates x and y, and on a scale which is determined by the base b. Obviously convenient calculation forms can be made for the computations. The expressions also seem to be suited for electronic computers.

Since we have assumed the pictures to be free from systematic errors, the condition $y'=y''$ will be fulfilled within the range of the accidental errors from the coordinate measurements. This condition obviously means a good check of systematic errors and also of the accidental errors.

In the same manner as is demonstrated above each pair of pictures can be treated. From identical points in the overlapping zone of the adjacent pairs of pictures, the scale and azimuth obviously can be transferred from pair to pair by simple coordinate transformation procedures. In this way the entire strip can be transformed into the same coordinate system. With the aid of at least two control points on the ground, the strip coordinates can then be transformed into the ground system. If superfluous control points are available, an adjustment procedure can be applied.

For the theory of errors of the radial triangulation procedure the differential formulae of (7) and (8) are of fundamental importance.

From differentiation of (7) and (8) we find after some simple rearrangements and with

b' is the difference $x'-x''$ or the base on the image scale

S is the scale factor of the images

x' and y' are the image coordinates in the left picture.

$$dx = \frac{x'}{b'}\,db - \left(\frac{x'}{b'} - 1\right) S dx' + \frac{x'}{b'} S dx'' + \frac{x'}{y'}\left(\frac{x'}{b'} - 1\right) S dy' - \frac{x'}{y'}\left(\frac{x'}{b'} - 1\right) S dy'' \tag{9}$$

$$dy = \frac{y'}{b'}\,db - \frac{y'}{b'} S dx' + \frac{y'}{b'} S dx'' + \frac{y'}{b'} S dy' - \left(\frac{x'}{b'} - 1\right) S dy'' \tag{10}$$

As soon as the errors dx', dy', dx'' and dy'' of the image coordinates are known, the influence upon the coordinates according to (7) and (8) can be found from the differential formulae (9) and (10). Radial errors have no influence upon the coordinates x and y. If all other known systematic errors of the image coordinates are corrected, the residual errors can be assumed as of accidental character. The propagation of such accidental errors to the coordinates x and y can then be found from (9) and (10), with the aid of the well-known (special) law of error propagation for such errors.

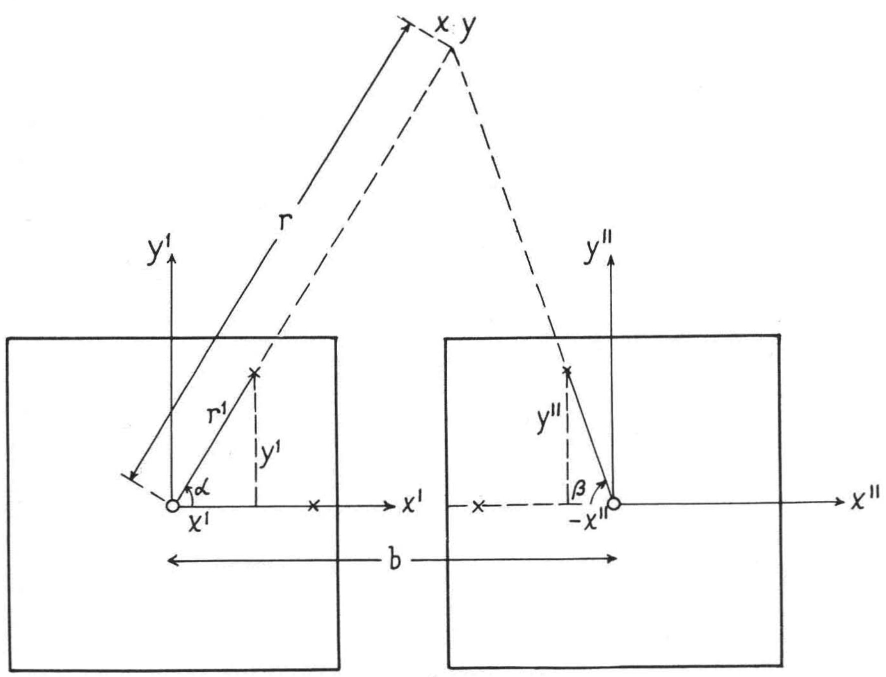


FIG. 1. The principles of intersection.

It is, however, important to determine the standard error of the image coordinates with respect to the measurements in a comparator or a coordinatograph, and with respect to the condition that the two images must be oriented along the same base. We therefore will give our attention to this special problem.

Determination of the Standard Errors of the Image Coordinates

We assume the image coordinates of all points of the image including the principal points are to be measured independently in a coordinatograph. The coordinate system of the coordinatograph is approximately identical with the image coordinate system.

For completeness sake we distinguish between the left and the right picture.

Left picture. The two coordinate systems are demonstrated in Figure 2. The image points are measured in the coordinate system u, v and will be transformed into the system $x'y'$ with the aid of the point $1'(x'=0,\ y'=0)$ and the *direction* through point $2'(x'=b',\ y'=0)$. The transformation consequently will be performed by two translations and one rotation.

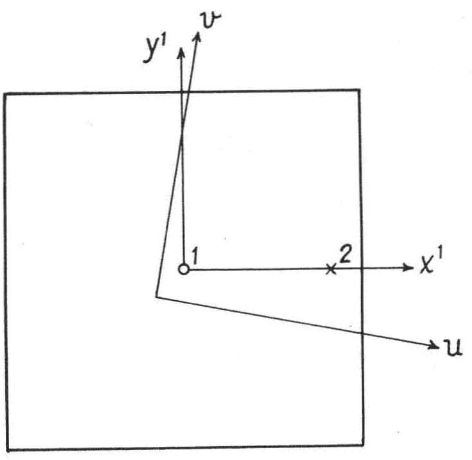


FIG. 2. The two coordinate systems u, v and x', y'. *Left image.*

The transformation formulae are:

$$du = du_0 - v\,d\kappa \tag{11a}$$

$$dv = dv_0 + u\,d\kappa \tag{11b}$$

du_0 and dv_0 are the translations and $d\kappa$ the rotation.

Since the coordinate systems are approximately identical we can use x' and y' respectively instead of v and u. Applied to the points $1'$ and $2'$ the formulae (11a) and (11b) become:

$$du_1 = du_0$$

$$dv_1 = dv_0$$

$$dv_2 = dv_0 + b'd\kappa$$

and consequently:

$$du_0 = du_1 \tag{12}$$

$$dv_0 = dv_1 \tag{13}$$

$$d\kappa = \frac{dv_2 - dv_1}{b'}. \tag{14}$$

The *corrections* to the measured coordinates u and v are obtained from (11) as:

$$du = -\,du_0 + v\,d\kappa \tag{15}$$

$$dv = -\,dv_0 - u\,d\kappa. \qquad 16)$$

The x'- and y'- coordinates are finally found as:

$$x' = u - du_0 + v\,d\kappa$$

$$= u - du_1 + \frac{v(dv_2 - dv_1)}{b'} \tag{17}$$

$$' = v - dv_0 - u\,d\kappa$$

$$= v - dv_1 - \frac{u(dv_2 - dv_1)}{b'}. \tag{18}$$

We assume all measurements of the u, v-coordinates to have equal accuracy or standard error μ. Consequently we can find the standard errors of x' and y' in applying the special law of error propagation to (17) and (18). Using x', y' instead of u, v in the coefficients we obtain after some rearrangements the standard errors:

$$m_{x'} = \frac{\mu}{b'}\sqrt{2(b'^2 + y'^2)} \tag{19}$$

$$m_{y'} = \frac{\mu}{b'}\sqrt{2(b'^2 + x'^2 - b'x')}. \tag{20}$$

The standard errors of arbitrary image coordinates $x'y'$ can now easily be determined.

The right picture. In exactly the same way the standard errors of the image coordinates $x''y''$ of the right picture can be found (Figure 3).

The expressions are:

$$m_{x''} = \frac{\mu}{b'}\sqrt{2(b'^2 + y''^2)} \tag{21}$$

$$m_{y''} = \frac{\mu}{b'}\sqrt{2(b'^2 + x''^2 + b'x'')} \tag{22}$$

The corresponding correlation-numbers are:

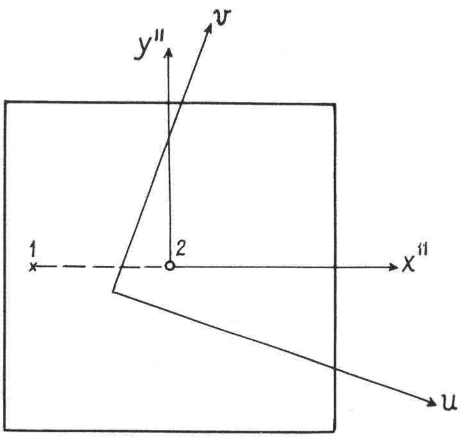


FIG. 3. The two coordinate systems u, v and x'', y''. *Right image.*

$$Q_{x'y'} = -\frac{2x'y'}{b'^2} + \frac{y}{b'} \tag{23}$$

$$Q_{x''y''} = -\frac{2x''y''}{b'^2} - \frac{y''}{b'} \tag{24}$$

CORRECTION OF SYSTEMATIC ERRORS OF THE IMAGE COORDINATES

The image coordinates are normally afflicted with different kinds of systematic errors. First of all are normally small deviations of the pictures from the vertical.

These deviations can, however, be determined most accurately from simple parallax observations, for instance with the mirror stereoscope and parallax bar. A rather complete description of the procedure has been given in [1]. After such a determination of the angles ϕ and ω for

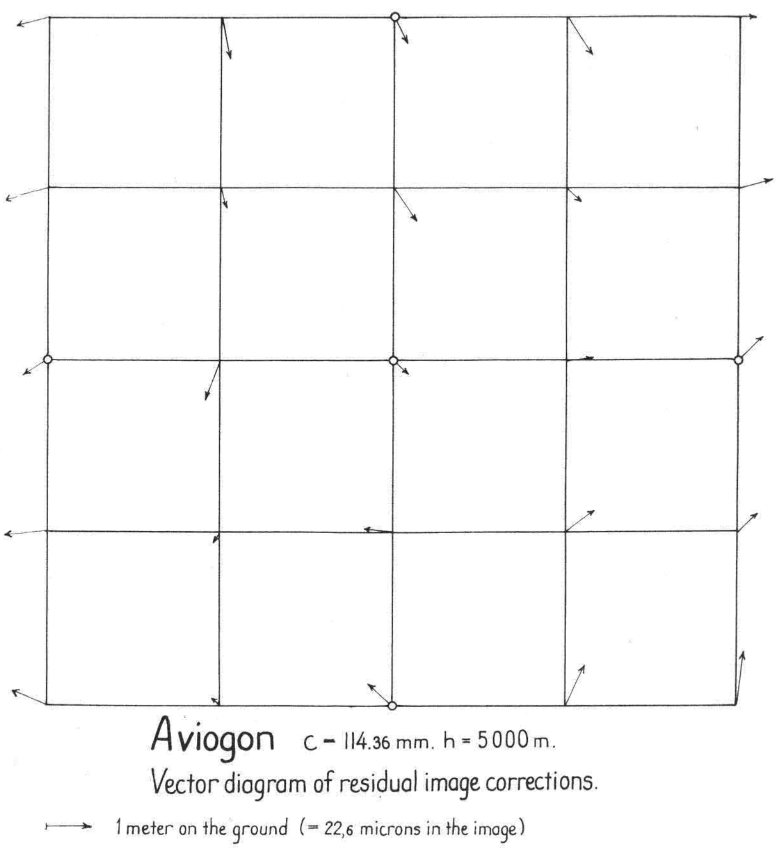


FIG. 4. Example of a vector diagram of residual image corrections after adjustment of the image coordinates in the five points which are indicated with small circles and after correction of the determined radial distortion. Swedish test-field. Film camera. The grid method. In this case the film shrinkage may have caused the major part of the residuals. From plate cameras or from repeated photography with film cameras the systematic errors of the lens can be reliably determined.

each picture—a kind of simplified aerial triangulation procedure can also be used as indicated in the mentioned paper—, the image coordinates can be corrected with respect to the inclination of the pictures.

Furthermore, there may be systematic errors of the image coordinates due to asymmetries of the lens system in the camera. Such asymmetries (affinities, tangential distortions etc.) can be determined under real photography conditions as indicated in [2].

From a vector diagram as reproduced in [2], see Figure 4, the image coordinates can be corrected.

After corrections as indicated above, the image coordinates can be regarded to be afflicted by mainly accidental errors. Radial distortion and similar radial errors are harmless in this connection.

Consequently, when the accidental errors are determined as standard errors of the image coordinate measurements, (see [3]), the error propagation of the radial triangulation can be studied in a reliable way by well-known methods.

Adjustment procedures according to the method of the least squares can be applied to the control-points discrepancies in a theoretically more correct manner than up to now. The accuracy following the adjustment can also be obtained from well-known procedures. The differential formulae (9) and (10) are of fundamental importance for these purposes.

Instruments for the Measurements

Obviously, the necessary measurements under normal conditions can be performed by instruments as simple as mirrcr stereoscopes and parallax bars, in addition to a good and stable single orthogonal comparator (for instance a coordinatograph). The systematic errors of the coordinate measurement device can be determined from the measurements of accurate grids. Then the measured image coordinates can easily be corrected for such systematic errors simultaneously with the corrections for other systematic errors as indicated above.

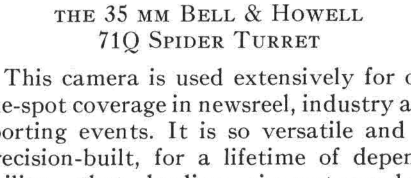
Of course a stereocomparator would be the most suitable instrument.

References

[1] Hallert, B.: "The Principles of Numerical Corrections in Aerial Photogrammetry." Photogrammetric Engineering, April 1956.

[2] ——— "The Grid Method and the *y*-Parallax Method for the Determination of Systematic Disturbances in Aerial and Terrestrial Photographs." *Svensk Lantmäteritidskrift*, Congress Number 1956.

[3] ——— "A New Method for the Determination of the Distortion and the Inner Orientation of Cameras and Projectors." *Photogrammetria* 1954–1955/3.



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