A Simplified Method of Locating the Point of Symmetry*†

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ABSTRACT: *A method is described whereby the magnitude and direction of the angle of camera tipping can be determined from analysis of the asymmetric values of the distortion for the case of objects lying at opposite but unequal angles from the central line of sight. The displacement of the central image from the principal point of autocollimation or point of symmetry is then readily determined. Theoretical analysis of the problem is given, and the results are confirmed with the aid of a set of experimental data contained in an article by Sewell describing the field method of calibration.*

1. INTRODUCTION

ONE of the problems that arises during the calibration of precision aerial mapping cameras is the accurate location of the proper principal point (1) with respect to the center of collimation (the point determined by the intersection of lines joining opposite pairs of collimation index markers). The principal point in photogrammetry is defined as the point at the foot of the perpendicular drawn from the interior perspective center to the plane of the photograph. The proper manner of locating this point has long been a subject of discussion. There are at present three different procedures that are employed in locating a point which is believed to be a sufficiently close approximation to the true principal point.

The first method, which has enjoyed considerable popularity, locates a point called the principal point of autocollimation (2, 3) which is the center of the image formed in the emulsion plane by the camera lens from an incident beam of parallel light which in the object space is perpendicular to the emulsion plane. This point has usually been designated "center cross" (4) by the author for convenience. This point is located by the simplest of the three methods and its popularity undoubtedly derives from this simplicity. To locate it, one needs only to aim an autocollimating telescope at a distant object. The camera is then interposed between the object and the telescope; a plane parallel plate with a mirror surface is pressed against the focal plane. With the aid of the autocollimating telescope, one can easily adjust the camera until its focal plane is normal to the line of sight of the telescope. The reflecting plate is then replaced by a photographic plate and the distant object is photographed. If the collimation index markers are simultaneously photographed, one can then easily locate the center cross with respect to the center of collimation.

The second method which was developed at this Bureau is based on the knowledge that most lenses suffer from small errors in centration that cause the lens to behave (in a first approximation) as though it was composed of a well-centered lens plus a thin prism (4, 5). This is further accentuated in the calibration of precision cameras by the presence of a filter in front of the lens that is seldom truly plane parallel which accordingly introduces a true prism effect. This prism effect causes a displacement of the center cross from the position

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it would have occupied in the absence of prism effect. This immediately poses the question as to which of these locations should properly be regarded as the principal point. It has been the custom at this Bureau to regard the point that would normally have been occupied by the center cross in the absence of prism effect, as the proper principal point of the photograph. It is an invariant point (12) when located under standard conditions of locating the center cross, and is not affected by subsequent changes in filters having different degrees of surface parallelism. The actual shift of the principal point from the center cross is easily determined from analysis of the asymmetric distortion pattern produced by such prism effect.

The third method came out of attempts to calibrate cameras by a field method (6). It was not customary nor readily practicable to aim the camera at the central target of a given target array with the aid of an autocollimating telescope, and consequently small errors of alignment sometimes were present. Such errors of alignment, called camera tipping, can produce an asymmetric distortion pattern similar but not identical to that produced by prism effect. In the analysis of the measurements made under these conditions of test, it was found possible to make the distortion pattern symmetrical by appropriate adjustment of the angles and image separations from the central image by computable increments. The point about which the distortion is tolerably symmetrical is called the point of symmetry.

In an earlier report (13), the author attempted to demonstrate, in the absence of prism effect, that all three methods located the same point. In that report, a method was described for the correction of the asymmetric distortion pattern arising from camera tipping. However, the study was limited by the requirement of having the comparable targets on opposite sides of the central target located at equal angles from the center. It is proposed to show in the present investigation that the methods developed may be used when the opposing angles are not equal.

2. Effect of Improper Alignment of Camera for Test

When the camera is being calibrated under conditions such that the focal plane of the camera is not normal to the line drawn to it from the central target, the calibration data are likely to show the presence of asymmetric distortion (7, 8, 9). This asymmetric distortion is sometimes confused with the asymmetric distortion produced by prism effect in the lens. It is, however, much simpler to eliminate the asymmetry of the distortion values arising from camera tipping than it is to eliminate or reduce the asymmetrical distortion arising from prism effect (4, 10). In this section, the effect of camera tipping upon the values of focal length and distortion is considered.

2.1 Determination of the Equivalent Focal Length

In Figure 1, N is the rear nodal point of a lens L, O is the point where a perpendicular dropped from N intersects the focal plane, consequently NO is equal to f, the equivalent focal length. The points of intersection with the focal plane of rays inclined at angles of β and α with the line NO in the object space are designated X and Y. The distances, OX and OY, are given in terms of β and α by the following expressions.

$$OX = f \tan \beta + D_\beta \tag{1}$$

and

$$OY = f \tan \alpha + D_\alpha \tag{2}$$

where D_β and D_α are the values of the radial distortion at points X and Y.

Let the camera now be rotated by amount ϵ about a line through N normal to the plane of the paper. The foot of the perpendicular from N moves from O to O' but the image of the central ray is formed at O'' at the intersection of line NO extended and the line $X'Y'$. The images formerly located at X and Y are now formed at X' and Y'. The relations connecting the various quantities and distances are

$$O'O'' = f \tan \epsilon, \tag{3}$$

$$O''X' = f \tan \epsilon + f \tan (\beta - \epsilon) + D_\beta' \tag{4}$$

$$O''Y' = f \tan (\alpha + \epsilon) - f \tan \epsilon + D_\alpha' \tag{5}$$

where $D_\beta' = D_{(\beta-\epsilon)}$ is the value of the distortion at X' and $D_\alpha' = D_{(\alpha+\epsilon)}$ is the value of the distortion at Y'. The small displacements, D_β' and D_α', are not indicated in the figure. The distances $O''X'$ and $O''Y'$ are the only quantities directly measurable. Adding eq. 4 and eq. 5, and bringing D_β' and D_α' to the left side of the expression we have

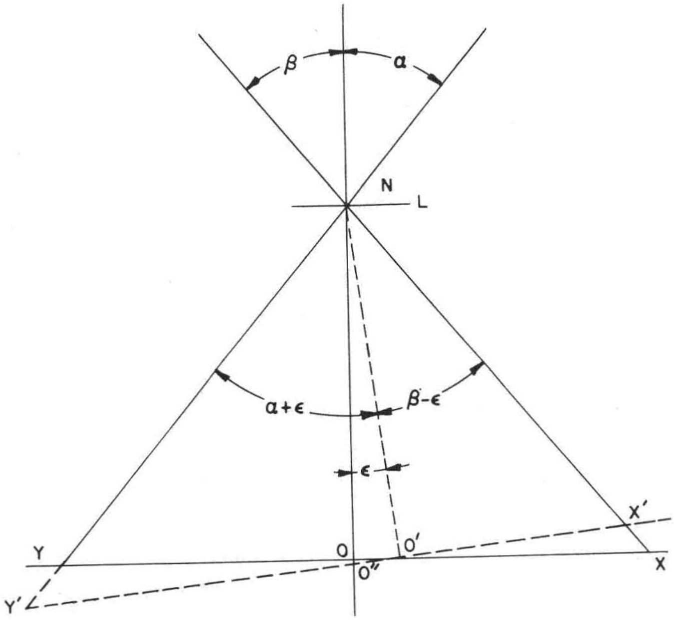

FIG. 1. Schematic drawing showing the image shift produced by rotating the camera by amount ϵ about an axis normal to the plane of the paper and passing through the rear nodal point of the lens L.

$$O''X' + O''Y' - (D_\beta' + D_\alpha') = f[\tan(\beta - \epsilon) + \tan(\alpha + \epsilon)] \quad (6)$$

which can be rewritten

$$O''X' + O''Y' - (D_\beta' + D_\alpha') = \frac{f \tan \alpha + \tan \beta)}{\cos^2 \epsilon[1 + (\tan \beta - \tan \alpha)\tan \epsilon - \tan \alpha \tan \beta \tan^2 \epsilon]}. \quad (7)$$

Solving for f, we have

$$f = \frac{Q(O''X' + O''Y') - (D_\beta' + D_\alpha')}{\tan \alpha + \tan \beta} \quad (8)$$

where

$$Q = \cos^2 \epsilon[1 + (\tan \beta - \tan \alpha) \tan \epsilon - \tan \alpha \tan \beta \tan^2 \epsilon] \quad (9)$$

For small ϵ, the term $(1-\epsilon^2)$ can be substituted for $\cos^2 \epsilon$, and ϵ for $\tan \epsilon$. Eq. 9 becomes

$$Q=1+(\tan \beta-\tan \alpha)\epsilon-\epsilon^2(1+\tan \alpha \tan \beta). \quad (10)$$

The act of determining the equivalent focal length, f, on the basis of measurements on a test negative has the effect of making the distortion zero at the point for which f is determined. For a determination of f based on two angular values β and α, the value of $(D_\beta'+D_\alpha')$ is therefore zero for $\beta=\alpha$; for $\beta \neq \alpha$, but not differing more than a few degrees, the term $(D_\beta'+D_\alpha')$ is only a few microns and can be neglected in the initial determination of f.

The likelihood of the differences in the values of the distortion between D_β' and D_α' is further reduced if one abides by the presently accepted rule (14) that the angles used in an equivalent focal length determination should not be greater than the angle subtended by the axial image and a point one-fifth the distance between the center and edge of the useful field. For a lens whose half-angular field is 45 degrees, this limits the angles, β and α, to approximately 11.3 degrees. At these relatively low values of the angle, the distortion changes only slightly with angle. The expression for f accordingly can be written

$$f = \frac{O''X' + O''Y'}{\tan \alpha + \tan \beta} [1 + \epsilon(\tan \beta - \tan \alpha) - \epsilon^2(1 + \tan \alpha \tan \beta)] \quad (11)$$

where ϵ is expressed in radians. It is interesting to note at this point that for $\beta=\alpha$, this equation becomes

$$f = \frac{O''X' + O''Y'}{2 \tan \beta} [1 - \epsilon^2(1 + \tan^2 \beta)]. \quad (12)$$

For $\beta \neq \alpha$, the terms in ϵ and ϵ^2 can usually be neglected provided ϵ is small, and $\tan \beta$

does not differ from tan α by more than 0.02. For example if $\epsilon=15$ minutes, $\beta=10°$ and $\alpha=9°$, the term within the brackets of eq. 11 has a value of 1.000059 which would lead to an error of +0.009 mm. in the determination of f. Accordingly for the usual conditions of test a useful value of f can be determined sufficiently closely by means of the approximate relation.

$$f=\frac{O''X'+O''Y'}{\tan\alpha+\tan\beta}. \tag{13}$$

While the value of f obtained with eq. 13 is sufficiently reliable to use in the calculations made to determine the value of ϵ which is described in a later section (see eq. 23), it is worthwhile to recompute f by using eq. 11 after ϵ has been determined. The value of f so determined with eq. 11 can be regarded as the true value of the equivalent focal length.

2.2 ASYMMETRIC DISTORTION PRODUCED BY IMPROPER ALIGNMENT.

When a camera is tipped in the manner described in section 2 asymmetric distortion is introduced by such tipping. This was treated for the case of a distortion-free lens in an earlier report (13). In this section, the magnitude of the asymmetric distortion will be derived in terms of β, α, and ϵ for the case of a lens having symmetric distortions arising from aberrations.

Distortion is usually computed with the aid of the formula

$$D_\beta = OX - f\tan\beta. \tag{14}$$

However, OX is not directly measurable, so $O''X'$ will be substituted for OX, and D_{T_1} for D_β, where D_{T_1} is the sum of the symmetric and asymmetric distortion, accordingly

$$D_{T_1} = f\tan\epsilon + \tan(\beta-\epsilon) + D_\beta' - f\tan\beta. \tag{15}$$

For small values of ϵ, it is substituted for tan ϵ. On dropping terms containing ϵ^3 or higher powers of ϵ, this expression becomes

$$D_{T_1} = -f\epsilon\tan^2\beta+\frac{f\epsilon^2\tan\beta}{\cos^2\beta}+D_\beta'. \tag{16}$$

In a similar manner, the distortion, D_{T_2}, on the opposite side of the center can be shown to be

$$D_{T_2} = f\epsilon\tan^2\alpha+\frac{f\epsilon^2\tan\alpha}{\cos^2\alpha}+D_\alpha'. \tag{17}$$

For a distortion-free lens, $D_\alpha'=D_\beta'=0$, and D_{T_2} is approximately equal to D_{T_1} but opposite in sign. For a lens having distortion, D_α' and D_β' are likely to be unequal in magnitude and generally but not always of the same sign. Consequently D_{T_2} is likely to differ markedly from D_{T_1} in magnitude and is frequently of the opposite sign. It is worthy of mention that, when one is analysing the results of measurement made on a camera whose direction of tipping is unknown, the sign or magnitude of the distortion values can serve as a guide in determining the angle of tipping and resultant direction of displacement of the central image O'' from the true position of the center cross or point of symmetry. The point of symmetry lies on the same side of the central image as those points showing maximum negative distortion. In those cases, where high positive distortion is inherent in the lens, it may be necessary to graph the values. Inspection of such a graph will readily show which branch is lower and which is higher because of the addition to the distortion values of the asymmetric contribution arising from camera tipping. In such instances, the point of symmetry lies on the side of the central image indicated by the lower branch of the curve. To express it another way, the central image is displaced from the point of symmetry toward those points showing maximum positive distortion or lie on the upper branch of the curve.

2.3 EVALUATION OF THE SYMMETRIC COMPONENT OF DISTORTION

The values of the distortion assigned to a camera-lens combination are usually obtained by averaging the values measured at the two corresponding opposite angular separations from the axis along a single diameter. A rough approximation of the values of the distortion at a given point whose angular separation from the axis is $(\beta+\alpha)/2$ where β does not depart from α by more than one degree is given by the expression

$$\frac{D_{T_1}+D_{T_2}}{2} = f\epsilon\left(\frac{\tan^2\alpha-\tan^2\beta}{2}\right)+\frac{f\epsilon^2}{2}\left(\frac{\tan\alpha}{\cos^2\alpha}+\frac{\tan\beta}{\cos^2\beta}\right)+\frac{D_\alpha'+D_\beta'}{2}. \tag{18}$$

For the case of $\beta=\alpha$, eq. 18 is

$$\frac{D_{T_1}+D_{T_2}}{2} = D_\beta+\frac{f\epsilon^2\tan\beta}{\cos^2\beta} \tag{19}$$

which shows that the average of the two asymmetric distortions is approximately equal to the actual value of the distortion for small values of ϵ. For larger values of ϵ, the term in ϵ^2 must be considered as has been shown in another article (13). While an approximate value of the distortion at point $(\beta+\alpha)/2$ can be obtained from eq. (18), more reliable determinations of D_β' and D_α' can be made after ϵ has been evaluated by using the procedures shown in the next section. For those angular regions where the distortion changes slowly with increasing angle, D_β' and D_α', can be regarded as equal to D_β and D_α for small values of ϵ.

For some lenses, there are regions where the distortion changes rapidly with angle, and corrections must be applied to arrive at more reliable values of the distortion D_β and D_α. It is known that the distortion pattern for a given type lens remains relatively invariant from lens to lens of the given type. Use can be made of this knowledge in determining a good value of the correction. Table 1 lists the values of the distortion for one type of wide angle lens in general use and Figure 2 shows the graph of the distortion values versus angle. These values of the distortion for this lens, whose focal length is approximately 150 mm. have been very carefully determined and can be considered accurate to within ± 0.003 mm. The change in distortion with angle is shown under the heading $\Delta D/\Delta\beta$ in Table 1 and is graphed in the lower part of Figure 2. It is evident that for $\epsilon \leqq 15$ minutes no serious error will result in considering $D_\beta = D_\beta'$ and $D_\alpha = D_\alpha'$ for the region from 0° to 37.5°. From 37.5° to 45° the value of $\Delta D/\Delta\beta$ is so large that corrections must be applied. This can be done

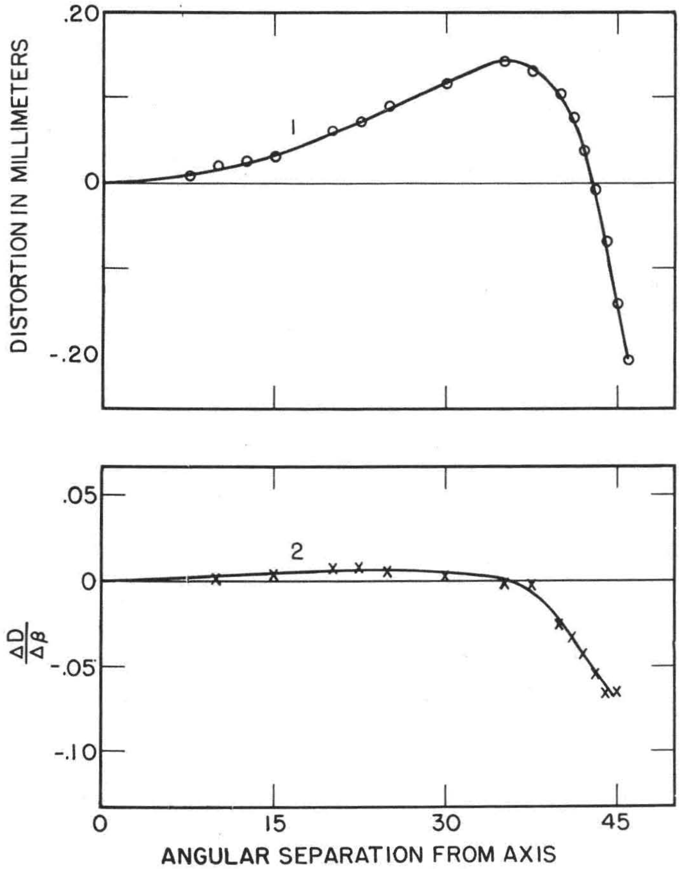

FIG. 2. Curve 1 shows variation of distortion, D_1 with angular separation from the axis, β, for a typical wide-angle lens. Curve 2 shows the rate of change of distortion with angle $(\Delta D/\Delta\beta)$ as a function of angular separation from the axis of the same lens.

TABLE 1

DISTORTION, D, REFERRED TO THE CALIBRATED FOCAL LENGTH VERSUS ANGULAR SEPARATION, β, FROM THE AXIS FOR A TYPICAL WIDE ANGLE LENS

The approximate change in distortion with β, $\Delta D/\Delta\beta$, is also shown for a selected set of values of β

β	D	$\Delta D/\Delta\beta$
degrees	mm.	mm./degrees
0	0.000	0.000
7.5	.008	
10.0	.020	0.003
12.5	.024	
15.0	.031	.004
20.0	.061	.007
22.0	.063	
22.5	.070	.009
23.0	.072	
24.5	.086	
25.0	.089	.005
25.5	.090	
29.5	.115	
30.0	.115	.004
30.5	.120	
34.5	.144	
35.0	.141	−.001
35.5	.143	
37.0	.126	
37.5	.130	−.002
38.0	.124	
39.5	.114	
40.0	.102	−.026
40.5	.088	
41.0	.075	−.032
42.0	.037	−.042
43.0	−.009	−.054
44.0	−.070	−.066
44.5	−.102	
45.0	−.141	−.067
45.5	−.170	
46.0	−.208	

sufficiently closely by means of the following relation

$$D_\beta = D_\beta' + \epsilon \frac{\Delta D}{\Delta \beta}. \qquad (20)$$

D_β' and D_α' can be readily evaluated with the aid of eq. 16 and eq. 17 when the magnitude and sign of ϵ is known. D_β and D_α can then be determined with the aid of eq. 20 when one has a good average distortion curve for the given type lens.

2.4 EVALUATION OF THE ANGLE OF CAMERA TIPPING

When a camera is tested under conditions such that the test targets are arranged along a line with the targets located at angles β and α on opposite sides of the central target, it is possible to evaluate ϵ in a relatively simple matter. To do so, it is only necessary to use the following relations,

$$D_{T_1} - D_{T_2} = -f\epsilon(\tan^2\beta + \tan^2\alpha) + f\epsilon^2\left(\frac{\tan\beta}{\cos^2\beta} - \frac{\tan\alpha}{\cos^2\alpha}\right) + (D_\beta' - D_\alpha'). \qquad (21)$$

For $\beta=\alpha$, the above expression becomes

$$D_{T_1} - D_{T_2} = -2f\epsilon\tan^2\beta \qquad (22)$$

from which ϵ can be readily determined (13). For $\beta\neq\alpha$ but not differing by more than 2°, the terms in $f\epsilon^2$ usually can be

TABLE 2

VALUES OF $\tan^2\beta$ AND $\tan\beta/\cos^2\beta$ FOR USE IN EVALUATING ERRORS IN DISTORTION

β	$\tan^2\beta$	$\tan\beta/\cos^2\beta$
degrees		
45	1.00000	2.0000
44	0.93258	1.8862
43	.86956	1.7434
42	.81702	1.6304
41	.75568	1.5262
40	.70409	1.4299
39	.65578	1.3408
38	.61043	1.2582
37.5	.58880	1.2189
37	.56791	1.1814
36	.52780	1.1100
35	.49028	1.0435
34	.45495	0.98138
33	.42172	.92328
32	.39050	.86885
31	.36108	.81778
30	.33339	.76980
29	.30725	.72462
28	.28270	.68202
27	.25959	.64181
26	.23785	.60376
25	.21744	.56770
22.5	.17159	.48524
20	.13250	.41218
15	.07182	.28718
10	.03108	.18181
7.5	.01733	.13394
5	.00766	.08816

TABLE 3

COMPARISON OF THE RELATIVE MAGNITUDES OF THE TERMS $A = -f \tan \epsilon\, (\tan^2\beta + \tan^2\alpha)$ AND $B = f \tan^2 \epsilon \left(\frac{\tan \beta}{\cos^2 \beta} - \frac{\tan \alpha}{\cos^2 \alpha}\right)$ FOR SEVERAL VALUES OF ϵ AND FOR $\beta = \alpha + 1°$ AND $\beta = \alpha + 2°$, FOR $f = 150$ MM.

	$\beta=\alpha+1°$		$\beta=\alpha+2°$	
	$\epsilon=10$ minutes		$\epsilon=10$ minutes	
β	A	B	A	B
degrees	mm.	mm.	mm.	mm.
45	−0.843	0.00017	−0.816	0.00033
40	− .593	.00011	− .574	.00022
35	− .412	.00008	− .398	.00015
30	− .279	.00006	− .269	.00011
	$\epsilon=20$ minutes		$\epsilon=20$ minutes	
45	−1.716	0.00068	−1.632	.00130
40	−1.187	.00045	−1.148	.00087
35	−0.825	.00032	−0.796	.00061
30	− .559	.00023	− .538	.00045
	$\epsilon=30$ minutes		$\epsilon=30$ minutes	
45	−2.532	0.00153	−2.449	.00293
40	−1.781	.00102	−1.722	.00196
35	−1.238	.00071	−1.195	.00138
30	−0.839	.00052	− .807	.00100

neglected, while the term $(D_{\beta}' - D_{\alpha})$ can be used either as a small correction factor or neglected if other errors present are sufficiently large as to obscure it. These assumptions can be readily verified with the aid of the information contained in Tables 1, 2, 3, and 4. Table 2 gives values of $\tan^2_{\beta}$ and $\tan \beta/\cos^2\beta$ for a series of values of β which are helpful in evaluating the relative magnitudes of the terms involving $f\epsilon$ and $f\epsilon^2$ in eq. 21. The relative magnitudes of these terms are shown in Table 3 for the case of $\beta = \alpha + 1°$ and $\beta = \alpha + 2°$ with values of ϵ of 10, 20, and 30 minutes for several values of β. Consideration of these tables shows that under these conditions the terms in $f\epsilon^2$ can be neglected. The value of $f\epsilon$ can accordingly be written

$$f\epsilon = -\frac{(D_{T_1} - D_{T_2})}{\tan^2\beta + \tan^2\alpha} + \frac{D_{\beta}' - D_{\alpha}'}{\tan^2\beta + \tan^2\alpha}. \quad (23)$$

Table 4 shows the differences in the measured values of distortion for a series of values of $\beta+\epsilon$ and $\beta-\epsilon$ where $\epsilon = 0.5°$, together with the correction to be made at angle β in the computed values of $f \tan \epsilon$. The values of $\Delta(f \tan \epsilon)$ will be equally valid in making corrections for the angle β and angle $\alpha = \beta - 1°$. It is clear that a fairly good correction can be made when the values of β and α are known, and the character of the average distortion pattern for the particular type lens is known. In

TABLE 4

CONTRIBUTIONS $\Delta(f \tan \epsilon)$ TO VALUES OF $f \tan \epsilon$ THAT WOULD ARISE FROM DIFFERENCES IN VALUES OF DISTORTION $D_{(\beta+\epsilon)}$ AND $D_{(\beta-\epsilon)}$ FOR A GIVEN ANGLE β FOR A COMMON TYPE OF WIDE ANGLE LENS WHEN THE CAMERA IS TIPPED THROUGH THE ANGLE $\epsilon = 0.5°$

β	$D_{(\beta-\epsilon)}$	$D_{(\beta-\epsilon)}$	$\Delta D/2$	$\Delta(f \tan \epsilon)$
degrees	mm.	mm.	mm.	mm.
45	−0.170	−0.102	−0.034	−0.034
42.5	− .009	.037	− .023	− .027
40	.008	.114	− .013	− .018
37.5	.124	.126	− .001	− .002
35	.143	.144	.000	.000
30	.120	.115	.002	.006
25	.090	.086	.002	.010
22.5	.072	.063	.004	.025

general when pairs of angles are selected such that $\beta - \alpha \leqq 1°$, the last term in eq. 23 can be neglected in the initial computations so we may use the following good approximation

$$f\epsilon = -\frac{(D_{T_1} - D_{T_2})}{\tan^2\beta + \tan^2\alpha} \tag{24}$$

3. Experimental Results for a Tipped Camera

In an earlier report, (13) the reliability of this method of evaluating the angle of camera tipping was established for the case of equal comparison angles, i.e. $\beta = \alpha$. The purpose of the present paper is to show that it is also reliable and relatively simple to use in the case of unequal comparison angles, i.e. $\beta \neq \alpha$. If this can be shown, then it would be applicable in the usual field-calibration operations. This laboratory has no experimental data upon which to make the necessary calculations as the test conditions here are for the case of $\beta = \alpha$. Fortunately, however, in a recent article by Sewell (6) an excellent set of measurements, made in the course of a field calibration of an airplane mapping camera, is presented.

It occurred to the author that possibly these measurements could be used to check the reliability of the proposed method of evaluating ϵ. The details of the test arrangements are completely described in the original article and will not be repeated here. Tables 5a and 5b show the angular separations in degrees of each numbered target from the central target, No. 67. The values of $\tan\beta$ and $\tan^2\beta$ are given for each target. (In these and subsequent tables, no distinction is made between β and α in the listing of constants to assist in the computation.) Tables 6a and 6b show the location of all images on the negative with respect to the central target. With the aid of eq. 13, the equivalent focal length f is found to be 154.060 mm. by using the measurements for targets Nos. 63 and 72. The distortion is then computed in the normal manner by using

$$D_2 = r - f \tan\alpha \tag{25}$$

for targets Nos. 36–66 and

$$D_1 = r - f \tan\beta \tag{26}$$

for targets Nos. 68–103.

The values of $f \tan\alpha$ and $f \tan\beta$ together with values of D_2 and D_1 are also listed in Tables 6a and 6b. The values of D are shown graphically in curve 1 of Figure 3. It is clear that the distortion pattern shows marked asymmetry.

It seems reasonable to assume that the major part of this asymmetry shown in curve 1 arises from camera tipping. It is therefore proper to use eq. 24 to determine the magnitude of the angle of camera tipping.

TABLE 5a

ANGULAR SEPARATIONS, α, IN THE OBJECT SPACE OF A SERIES OF TARGETS FROM THE CENTRAL TARGET (NO. 67)

Values of $\tan\alpha$ and $\tan^2\alpha$ are also given. The values of α are derived from information contained in table 5, p. 168, of MANUAL OF PHOTOGRAMMETRY (2nd Ed.)

Target No.	α	$\tan\alpha$	$\tan^2\alpha$
	degrees		
35	44.3239	0.9766739	0.9538919
37	43.5050	.9491305	.9008487
38	42.6442	.9209718	.8481890
39	41.7928	.8938771	.7990163
40	40.9236	.8669484	.7515995
41	40.0389	.8402572	.7060322
42	39.1242	.8133794	.6615862
43	39.2025	.7869930	.6193580
44	37.2611	.7607235	.5787002
45	36.2839	.7341405	.5389623
46	35.2903	.7077853	.5009600
47	34.2756	.6815299	.4644830
48	33.2444	.6554890	.4296658
49	32.2144	.6300847	.3970067
50	31.1586	.6046344	.3655828
51	30.0608	.5787660	.3349701
52	28.9772	.5537890	.3066823
53	27.8389	.5281082	.2788983
54	26.7050	.5030570	.2530663
55	25.5492	.4780300	.2285127
56	24.3933	.4534791	.2056433
57	23.1939	.4284745	.1835904
58	19.5300	.3547079	.1258177
59	18.2894	.3305132	.1092390
60	17.0444	.3065782	.0939902
61	11.9403	.2114678	.0447186
62	10.6175	.1874610	.0351416
63	9.3172	.1640646	.0269172
64	7.9953	.1404571	.0197282
65	2.6800	.0468090	.0021911
66	1.3383	.0233620	.0005458
67	0.0000	.0000000	.0000000

TABLE 5b

(CONTINUATION OF 5a) ANGULAR SEPARATIONS IN THE OBJECT SPACE FROM THE CENTRAL TARGET (NO. 67)

Values of tan β and $\tan^2 \beta$ are also given

Target No.	β	tan β	$\tan^2 \beta$
	degrees		
67	0.0000	0.0000000	0.0000000
68	3.9986	.0699023	.0048863
69	5.3278	.0932566	.0086968
70	6.6753	.1170360	.0136974
71	7.9930	.1404163	.0197167
72	9.3078	.1638961	.0268619
73	10.6192	.1874918	.0351532
74	11.9319	.2113145	.0446538
75	13.2144	.2348131	.0551372
76	14.4811	.2582657	.0667012
77	16.9914	.3055665	.0933709
78	18.2406	.3295686	.1086155
79	19.4556	.3532467	.1247832
80	20.6619	.3771088	.1422110
81	21.8503	.4009902	.1607931
82	23.0336	.4251671	.1807671
83	24.1939	.4492899	.2018614
84	25.3250	.4732318	.2239483
85	26.4439	.4973597	.2473667
86	27.5422	.5215035	.2719659
87	28.6203	.5456774	.2977638
88	30.7158	.5941296	.3529900
89	31.6539	.6165106	.3800742
90	32.7425	.6430366	.4134961
91	33.7153	.6673004	.4452898
92	34.6622	.6914573	.4781132
93	35.6039	.7160327	.5127028
94	36.5180	.7404474	.5482623
95	37.4144	.7649561	.5851578
96	38.2967	.7896589	.6235612
97	39.1472	.8140467	.6626720
98	39.9839	.8386209	.7032850
99	40.8072	.8633962	.7454530
100	41.6009	.8878665	.7883069
101	42.3825	.9125656	.8327760
102	43.1461	.9372938	.8785197
103	43.8975	.9622375	.9259010

$$f\epsilon = -\frac{(D_{T_1} - D_{T_2})}{\tan^2 \beta + \tan^2 \alpha} \text{ or } \frac{(D_2 - D_1)}{\tan^2 \beta + \tan^2 \alpha} \cdot \quad (24)$$

It is advantageous to select target pairs for which β and α differ as little as possible. This has been done and the determination of $f\epsilon$ (or $f \tan \epsilon$) is shown in Table 7 for 29

TABLE 6a

VALUES OF DISTANCES, r, SEPARATING THE IMAGES AT NUMBERED LOCATIONS FROM THE CENTRAL IMAGE (LOCATION NO. 67)

Values of $f \tan \alpha$, and $D_2 = r - f \tan \alpha$, are also shown. The data are derived from information contained in table 6, p. 169 of the MANUAL OF PHOTOGRAMMETRY (2nd Ed.). $f = 154.060$ mm.

Target No.	r	$f \tan \alpha$	D_2
	mm.	mm.	mm.
36	150.902	150.466	+0.436
37	146.706	146.223	.483
38	142.368	141.885	.517
39	138.209	137.711	.498
40	134.078	133.562	.516
41	129.963	129.450	.513
42	125.786	125.309	.477
43	121.728	121.244	.484
44	117.650	117.197	.453
45	113.538	113.102	.436
46	109.462	109.041	.421
47	105.375	104.996	.379
48	101.336	100.985	.351
49	97.412	97.071	.341
50	93.463	93.150	.313
51	89.448	89.165	.283
52	85.585	85.317	.268
53	81.608	81.860	.248
54	77.728	77.501	.227
55	73.848	73.645	.203
56	70.048	69.863	.185
57	66.172	66.011	.161
58	54.763	54.646	.117
59	51.008	50.919	.089
60	47.306	47.231	.075
61	32.614	32.579	.035
62	28.898	28.880	.018
63	25.292	25.276	.016
64	21.651	21.639	.012
65	7.220	7.211	.009
66	3.613	3.599	.014
67	0.000	0.000	.000

target pairs. The approximate average value of $(\beta+\alpha)/2$ is listed so that the angular region occupied by a given target pair is readily apparent. Values of $\tan^2\beta+\tan^2\alpha$ are derived from Table 5. The column headed D_2-D_1 is simply the difference in the measured values of the distortion for

TABLE 6b

(CONTINUATION OF 6a) VALUES OF DISTANCES, r, SEPARATING THE IMAGES AT NUMBERED LOCATIONS FROM THE CENTRAL IMAGE (LOCATION NO. 67)

Values of $f \tan \beta$ and $D_1 = r - f \tan \beta$ are also shown

Target No.	r	$f \tan \beta$	D_1
	mm.	mm.	mm.
67	0.000	0.000	0.000
68	10.772	10.769	.003
69	14.362	14.367	−.005
70	18.026	18.031	−.005
71	21.622	21.632	−.010
72	25.234	25.250	−.016
73	28.867	28.885	−.018
74	32.536	32.555	−.019
75	36.144	36.175	−.031
76	39.757	39.788	−.031
77	47.037	47.076	−.039
78	50.732	50.773	−.041
79	54.380	54.421	−.041
80	58.049	58.097	−.048
81	61.732	61.776	−.044
82	65.444	65.501	−.057
83	69.151	69.213	−.067
84	72.847	72.906	−.059
85	76.530	76.623	−.093
86	80.238	80.343	−.105
87	83.968	84.067	−.099
88	91.412	91.532	−.120
89	94.849	94.978	−.129
90	98.927	99.066	−.139
91	102.645	102.804	−.159
92	106.358	106.526	−.168
93	110.109	110.312	−.203
94	113.857	114.073	−.216
95	117.610	117.849	−.239
96	121.387	121.655	−.268
97	125.107	125.412	−.315
98	128.855	129.198	−.343
99	132.625	133.015	−.390
100	136.335	136.785	−.450
101	140.087	140.590	−.503
102	143.844	144.399	−.555
103	147.632	148.242	−.610

the members of a given target pair. To assure ease of comprehension it may be mentioned that D_2 corresponds to D_{T_2} of eq. 17 and D_1 corresponds to D_{T_1} of eq. 16 and consequently $D_2 - D_1 = -(D_{T_1} - D_{T_2})$. The column headed $f \tan \epsilon$ gives the values of this quantity determined for each target pair.

Although the lens is not distortion-free, the values of $f \tan \epsilon$, shown in Table 7, do not appear to vary from one another in any consistent manner. Consequently one may assume that any single value is as valid as any other single value, and that the best value is the average of all 29 single values, which is $\overline{f \tan \epsilon} = 0.596$ mm. The probable error of the mean is ± 0.003 mm., and the probable error of a single value is ± 0.014 mm. The value of ϵ is readily computed and is equal to 0.003869 radians or 13.31 minutes of arc. It is of course realized that the small differences in distortion between β and α should be reflected by differences in the values of $f \tan \epsilon$, and are calculable with the aid of the information given in Tables 1 and 4.

With this in mind the values of $\Delta \bar{a}$, the departure of a single value from the mean, are listed in Table 8 for each target-pair. In addition the values of $(\alpha+\beta)/2$ and $\alpha-\beta$ are listed for aid in any computation necessary. It is evident, from consideration of this table that there is little to be gained by making a distortion correction. For example, for target-pair 36 and 103, we might expect a distortion of 0.024 which would produce a systematic error of approximately 0.013 mm., which is about $\frac{1}{3}$ of the −.040 mm. observed for $\Delta \bar{a}$. On the other hand for patterns 38 and 101, the distortion difference would be 0.012 corresponding to a systematic error of 0.007 or about $\frac{2}{3}$ of $\Delta \bar{a}$. But because of the sign reversed in the two cases, the change in $\Delta \tan \epsilon$ from distortion is 0.005 mm. while the actual $\Delta \tan \epsilon$ is 0.051 mm. It is therefore reasonable to state that the other uncertainties outweigh the uncertainty arising from distortion so that in the present case the error arising from distortion can be neglected. In addition, further compensation may be expected from the fact that $\beta - \alpha$ does not have the same sign throughout. The possible corrections for distortion are both positive and negative and when the average is taken over such a wide range of angles, they may be regarded as negligible.

3.1 LOCATION OF "POINT OF SYMMETRY"

In the absence of prism effect, $f \tan \epsilon$ is the distance separating the principal

TABLE 7

COMPUTATION OF f TAN ϵ OR DISPLACEMENT OF THE POINT OF SYMMETRY INFERRED FROM THE ASYMMETRIC DISTORTION VALUES GIVEN IN TABLE 6

Target Pairs	$(\beta+\alpha)/2$	$\tan^2\beta+\tan^2\alpha$	D_2-D_1	$f\tan\epsilon$
	degrees		mm.	mm.
36 & 103	44.1	1.8798	1.046	0.556
37 & 102	43.3	1.7794	1.038	.583
38 & 101	42.5	1.6810	1.020	.607
39 & 100	41.7	1.5873	0.948	.597
40 & 99	40.9	1.4970	.906	.605
41 & 98	40.0	1.4093	.856	.607
42 & *97*	39.1	1.3243	.792	.598
43 & *96*	38.2	1.2429	.752	.605
44 & *95*	37.3	1.1638	.692	.595
45 & *94*	36.4	1.0872	.652	.600
46 & *93*	35.4	1.0137	.624	.616
47 & *92*	34.5	0.94260	.547	.580
48 & *91*	33.5	.87496	.510	.583
49 & *90*	32.5	.81050	.480	.592
50 & *89*	31.4	.74566	.442	.593
51 & *88*	30.4	.68796	.403	.586
52 & 87	28.8	.60445	.367	.607
53 & 86	27.7	.55086	.353	.641
54 & 85	26.6	.50043	.320	.639
55 & 84	25.4	.45246	.262	.579
56 & 83	24.3	.40750	.252	.618
57 & *82*	23.3	.36436	.218	.598
58 & 79	19.5	.25060	.158	.630
59 & 78	18.3	.21785	.130	.597
60 & 77	17.0	.18736	.114	.608
61 & 74	11.9	.089372	.054	.604
62 & *73*	10.6	.070295	.036	.512
63 & 72	9.3	.053779	.032	.595
64 & 71	8.0	.039445	.022	.558

Average for $n=29$ is 0.596 mm.
$PE_m=\pm 0.003$ mm.
$PE_s=\pm 0.014$ mm.

point of autocollimation and the image of the central target. The shift of the central image with respect to the focal plane coordinates is in the direction of maximum positive distortion. Accordingly in the present instance, the principal point of autocollimation is located 0.596 mm. distant from the central image and lies on the same side of it as the image of target No. 103. It can be stated that generally the "point of symmetry" or "principal point of autocollimation," (6,11) lies on the side of maximum negative distortion.

3.2 DETERMINATION OF DISTORTION FOR TIPPED CAMERA

After the value of f tan ϵ has been established, the distortion can be readily evaluated with the aid of eq. 16 and eq. 17, which can be rewritten as follows:

$$D_\alpha' = D_{T_2} - f\epsilon\tan^2\alpha - \frac{f\epsilon^2\tan\alpha}{\cos^2\alpha} \qquad (27)$$

and

$$D_\beta' = D_{T_1} + f\epsilon\tan^2\beta - \frac{f\epsilon^2\tan\beta}{\cos^2\beta}. \qquad (28)$$

On substituting the measured values for $f\epsilon$ and $f\epsilon^2$, these become

$$D_\alpha' = D_{T_2} - 0.596\tan^2\alpha - \frac{0.0024\tan\alpha}{\cos^2\alpha} \qquad (29)$$

and

TABLE 8

COMPARISON OF THE DEPARTURE FROM THE AVERAGE $\Delta\bar{a}=f \tan \epsilon - \overline{f \tan \epsilon}$ WITH THE DIFFERENCE $\alpha-\beta$ FOR EACH TARGET PAIR USED IN DETERMINING THE AVERAGE $f \tan \epsilon = 0.596$ mm.

Target Pairs	$(\alpha+\beta)/2$	$\alpha-\beta$	$\Delta\bar{a}$
	degrees	degrees	mm.
36 — 103	44.1	+0.43	−0.040
37 — 102	43.3	.36	− .013
38 — 101	42.5	.26	.011
39 — 100	41.7	.19	.001
40 — 99	40.9	.12	.009
41 — 98	40.0	.03	.011
42 — 97	39.1	− .02	.002
43 — 96	38.2	− .09	.009
44 and 95	37.3	− .15	− .001
45 and 94	36.4	− .23	.004
46 and 93	35.4	− .31	.020
47 and 92	34.5	− .39	− .016
48 and 91	33.5	− .47	− .017
49 and 90	32.5	− .53	− .004
50 and 89	31.4	− .50	− .003
51 and 88	30.4	− .66	− .010
52 and 87	28.8	− .36	.011
53 and 86	27.7	− .30	.045
54 and 85	26.6	− .26	.043
55 and 84	25.4	− .22	− .017
56 — 83	24.3	− .20	.022
57 — 82	23.3	− .16	.002
58 — 79	19.5	− .07	.034
59 — 78	18.3	− .05	.001
60 — 77	17.0	− .05	.012
61 — 74	11.9	− .01	.008
62 — 73	10.6	.00	− .074
63 — 72	9.3	− .01	− .001
64 — 71	8.0	.00	− .038

$$D_{\beta}' = D_{T_1} + 0.596 \tan^2 \beta - \frac{0.0024 \tan \beta}{\cos^2 \beta} \cdot \quad (30)$$

The last term in the above equations may be carried or dropped as warranted by consideration of its magnitude. In the present instance it contributes a maximum of −0.005 mm. at 45°. However, it was dropped in the remaining calculations although it can readily be evaluated with the aid of Table 3. Tables 9a and 9b show in column 2, the value of 0.596 $\tan_{\alpha}^2$ and 0.596 $\tan_{\beta}^2$ for each value of α and β; column 3 shows the actual values of D_{α}' and D_{β}'. The "$\tan^2\alpha$" correction is plotted as curve 2 in Figure 3, and the values of D_{α}' and D_{β}' are plotted as curve 3 in the same figure. It is clear that the values of D_{β}' and D_{α}' are now quite symmetrical in value. If warranted, further corrections can be made to determine the actual values

TABLE 9a

ADJUSTMENT OF ASYMMETRIC VALUES OF DISTORTION (SEE TABLE 6a) TO COMPENSATE FOR EFFECTS OF CAMERA TIPPING

Column 1 shows initial asymmetric values of distortion D_2, column 2 shows the values of $f \tan \epsilon \tan^2 \alpha$ (contribution from camera tipping), and column 3 shows the adjusted values of distortion ($D_{\alpha}' = D_2 - f \tan \epsilon \tan^2 \alpha$) for $f=154.060$ mm. and $f \tan \epsilon = +0.596$ mm.

Target No.	α	1	2	3
	degrees	mm.	mm.	mm.
36	44.3239	+0.436	0.568	−0.132
37	43.5050	.483	.537	− .054
38	42.6442	.517	.506	+ .011
39	41.7928	.498	.476	.022
40	40.9236	.516	.448	.068
41	40.0389	.513	.421	.092
42	39.1242	.477	.394	.083
43	38.2025	.484	.369	.115
44	37.2611	.453	.345	.108
45	36.2839	.436	.321	.115
46	35.2903	.421	.299	.122
47	34.2756	.379	.277	.102
48	33.2444	.351	.256	.095
49	32.2144	.341	.237	.104
50	31.2586	.313	.218	.095
51	30.0608	.283	.200	.083
52	28.9772	.268	.183	.085
53	37.8389	.248	.166	.082
54	26.7050	.227	.151	.076
55	25.5492	.203	.136	.067
56	24.3933	.185	.123	.063
57	23.1939	.161	.109	.052
58	19.5300	.117	.075	.042
59	18.2894	.089	.065	.024
60	17.0444	.075	.056	.019
61	11.9403	.035	.027	.008
62	10.6175	.018	.021	−.003
63	9.3172	.016	.016	.000
64	7.9953	.012	.012	.000
65	2.6800	.009	.001	.008
66	1.3383	.014	.000	.014
67	0.0000	.000	.000	.000

TABLE 9b

ADJUSTMENT OF ASYMMETRIC VALUES OF DISTORTION (SEE TABLE 6b) TO COMPENSATE FOR CAMERA TIPPING

Column 1 shows initial asymmetric values of distortion, D_1, column 2 shows the values of f tan ϵ tan^2 β (contribution from camera tipping), and column 3 shows the adjusted values of distortion, $(D_\beta' = D_2 - f \tan \epsilon \tan^2 \beta)$ for $f = 154.060$ mm. and $f \tan \epsilon = +0.596$ mm.

Target No.	β	1	2	3
	degrees	mm.	mm.	mm.
67	0.0000	0.000	0.000	0.000
68	3.9986	−.003	+.003	+.006
69	5.3278	−.005	.005	.000
70	6.6753	−.005	.008	.003
71	7.9930	−.010	.012	.002
72	9.3078	−.016	.016	.000
73	10.6192	−.018	.021	.003
74	11.9319	−.019	.026	.007
75	13.2144	−.031	.033	.002
76	14.4811	−.031	.040	.009
77	16.9914	−.039	.056	.017
78	18.2406	−.041	.065	.024
79	19.4556	−.041	.074	.033
80	20.6619	−.048	.085	.037
81	21.8503	−.044	.096	.052
82	23.0336	−.057	.108	.051
83	24.1939	−.067	.120	.053
84	25.3250	−.059	.133	.074
85	26.4439	−.093	.147	.054
86	27.5422	−.105	.162	.057
87	28.6203	−.099	.177	.078
88	30.7158	−.120	.210	.090
89	31.6539	−.129	.226	.097
90	32.7425	−.139	.246	.107
91	33.7153	−.159	.265	.106
92	34.6622	−.168	.285	.117
93	35.6039	−.203	.306	.103
94	36.5180	−.216	.327	.109
95	37.4144	−.239	.349	.110
96	38.2967	−.268	.372	.104
97	39.1472	−.315	.395	.080
98	39.9839	−.343	.419	.076
99	40.8072	−.390	.444	.054
100	41.6008	−.450	.470	.020
101	42.3825	−.503	.496	−.007
102	43.1461	−.555	.524	−.031
103	43.8975	−.610	.552	−.058

of D_β and D_α if ϵ is sufficiently large that appreciable variations can be expected. Such corrections can be made easily with the aid of Tables 1 and 4.

4. CONCLUSION

It is evident from the foregoing theory and experimental verification that the "tipped camera" method of analysis can be applied in cases where the members of pairs of corresponding opposite angles are unequal in magnitude. It can therefore be applied in the case of camera calibration by the usual field methods. It has the advantage of simplifying the calculations in that after the values of α and β have been determined for the camera station, the same values of tan α, tan$^2\alpha$, tan $\alpha/2$, etc., can be used repeatedly for as many cameras as are tested.

In addition, this study clearly establishes that, in the absence of prism effect, the "point of symmetry" is identical with the "principal point of autocollimation." It can be stated further that, in the presence of prism effect, the "point of symmetry" is definitely neither the "principal point of autocollimation" nor the true principal point: it is, however, the point about which the distortion is minimized for some particular angle and undoubtedly is a good compromise. It is likely that further study along these lines may be worthwhile.

5. ACKNOWLEDGMENTS

This experimental verification, contained in this paper, was made easy by the ready availability of the excellent set of measurements contained in the article "Field Calibration of Aerial Mapping Cameras" by Eldon .Sewell, (6) pp. 137–177, Manual of Photogrammetry, Second Edition, Published by the American Society of Photogrammetry (1952). The excellent agreement between the theory, herein developed, and the measurements reported by Sewell indicates that the measurements must have been made with painstaking care and have high reliability.

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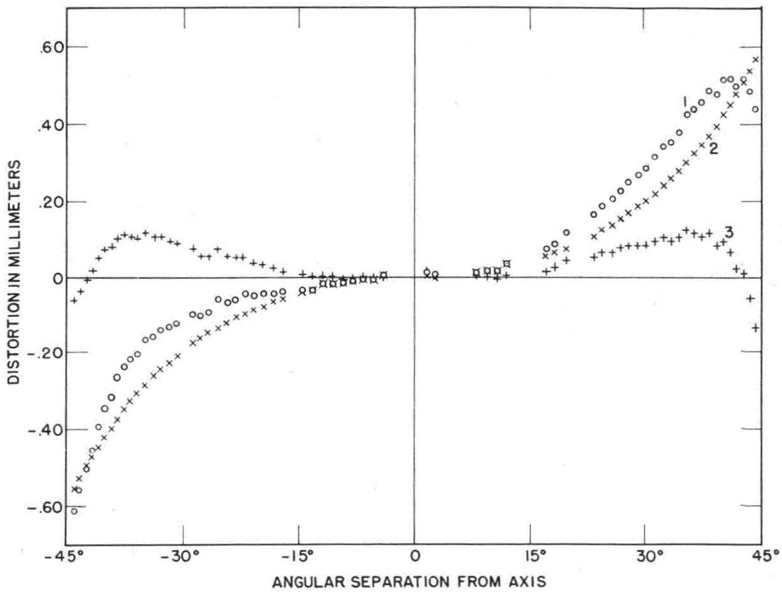

FIG. 3. Adjustment of asymmetric distortion arising from camera tipping. The circles (curve 1) show the actual values of distortion as initially determined from the measurements. The X's (curve 2) show the asymmetric distortion produced by a tipping of amount $\epsilon = 0.003869$ radians or 13.31 minutes. The crosses (curve 3) show the symmetrical distortion pattern remaining when curve 2 is subtracted from curve 1.

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