*Prism Effect, Camera Tipping, and Tangential Distortion**†

FRANCIS E. WASHER,
National Bureau of Standards,
Washington, D. C.

ABSTRACT: *The compensation of the asymmetric distortion arising from prism effect by the introduction of camera tipping is discussed. Complete neutralization of the asymmetric distortion is possible at a selected angular separation from the axis; at other angular separations the magnitude of the asymmetric distortion may be reduced. It is shown that tangential distortion may be produced by prism effect. The magnitude of the radial asymmetric distortion produced by prism is shown to be greater than the magnitude of the tangential distortion produced by the same prism. Theoretical analysis and experimental examples of the relation connecting tangential and radial asymmetric distortion produced by prism are shown.*

1. INTRODUCTION

THE final accuracy of a finished map produced by photogrammetric processes is dependent upon the accuracy of the equipment used in the various steps intervening between the initial photography and the final map. To insure highest accuracy, the various optical devices used in each step are carefully calibrated. Calibration of precision mapping cameras used in the initial photography has been performed for many years at the National Bureau of Standards. To perform these calibrations specialized equipment has been developed and constructed (1, 2). Methods of analysis have been devised for interpreting the results of measurement (3, 4) and extensive studies of the factors influencing the accuracy of calibration have been made.

In a recent series of papers, the author has discussed sources of error in camera calibration (5, 6), camera tipping (7), point of symmetry (8), and prism effect (9). In this paper, a comparison is made of the two types of asymmetric radial distortion, that produced by tipping and that produced by prism effect.[1] In addition some brief notes on the effect of lens tipping are included. The subject of tangential distortion produced by prism effect is discussed in some detail. It is found that the tangential distortion is smaller in magnitude than radial asymmetric distortion produced by prism effect.

2.0 COMPENSATION OF PRISM EFFECT BY CAMERA TIPPING

In previous papers (7, 8, 9) it has been shown that radial asymmetric distortion can arise from either of two causes, prism effect or camera tipping. It is, therefore, worthwhile to consider these two effects

[1] The term "prism effect" as used throughout this paper is intended to refer to those effects that are produced by placing a prism in front of the camera lens or are produced by lens decentration. Other expressions synonymous to "prism effect" are (1) prismatic effect, (2) prismatical effect, (3) equivalent prism, (4) effective prism, and (5) effect of prism. The fact that a decentered lens can be treated as a combination of a lens plus a prism is discussed in texts (20, 21, 22) dealing with spectacle lenses.

* This is the fourth paper of a series dealing with problems that relate to the calibration of precision airplane mapping cameras. The first paper, entitled "Sources of Error in Various Methods of Airplane Camera Calibration," is included in the September, 1956 issue (Vol. XXII, no. 4). The second is in the March 1957 issue (Vol. XXIII, no. 1); the title is "A Simplified Method of Locating the Point of Symmetry." The third—"The Effect of Prism on the Location of the Principal Point"—is in the June 1957 issue (Vol. XXIII, no. 3).

† This work was sponsored in part by the U. S. Air Force.

together in order to note their similarities and differences, and to determine whether or not the radial asymmetric distortion produced by prism effect can be neutralized or at least compensated in part by camera tipping. To this end, the pertinent equations are herein repeated to facilitate comparison.

In the six equations that follow, subscripts not present in the original papers have been introduced to distinguish between radial asymmetric distorition, D_T, produced by camera tipping and radial asymmetric distortion, D_p, produced by prism effect. For the case of camera tipping (7, 8),

$$D_{T_1} = -f\epsilon_T \tan^2\beta + \frac{f\epsilon^2_T \tan\beta}{\cos^2\beta} \tag{1}$$

$$D_{T_2} = f\epsilon_T \tan^2\beta + \frac{f\epsilon^2_T \tan\beta}{\cos^2\beta} \tag{2}$$

and

$$D_{T_1} - D_{T_2} = -f\epsilon_T \tan^2\beta \tag{3}$$

where D_{T_1} and D_{T_2} are the values of the asymmetric distortion in the image plane, for coplanar rays inclined at angle β to the line joining the front nodal point of the lens and central target in the object space, induced by tipping the camera through the angle ϵ_T from the initial correct position of alignment for calibration. D_{T_1} is the value of the distortion in the image plane for an object located at angle β from the axis on one side of the axis (or $+\beta$), and D_{T_2} is the corresponding value for an object located at angle β from the axis on the opposite side of the axis (or $-\beta$).

It is of course possible to describe the entire distortion pattern by means of Eq. (1) alone by following the usual sign conventions. However, for all practical purposes it is more convenient to treat β as being positive in all computations and to use Eq. (2) for those situations where β could be regarded as negative. The expression given in Eq. (3) is simply the difference between Eq. (1) and Eq. (2); it is useful in evaluating ϵ_T when D_{T_1} and D_{T_2} are known; it may also be regarded as giving the approximate magnitude of the major component of average asymmetric distortion induced by camera tipping.

For the case of prism effect (9),

$$D_{P_1} = \frac{f\epsilon_P}{\cos^2\beta}(1+\epsilon_P \tan\beta) - f\epsilon_0 \tag{4}$$

$$D_{P_2} = \frac{-f\epsilon_P}{\cos^2\beta}(1-\epsilon_P \tan\beta) + f\epsilon_0 \tag{5}$$

and

$$\frac{D_{P_1} - D_{P_2}}{2} = f\epsilon_0\left[\frac{\epsilon_P}{\epsilon_0 \cos^2\beta} - 1\right] \tag{6}$$

where D_{P_1} and D_{P_2} are the values of the asymmetric distortion in the image plane for coplanar rays inclined at angle β to a line normal to the focal plane of the camera and passing through a distant target in the object space. The angle ϵ_P is the deviation produced by prism effect and ϵ_0 is the prism deviation at $\beta = 0°$. The axis of the prism effect is normal to the plane containing the incident rays inclined at angle β to the system axis as above defined. An additional useful formula is

$$\epsilon_P = \alpha\left[\frac{\sqrt{n^2 - \sin^2\beta}}{\cos\beta} - 1\right] \tag{7}$$

where α is the angle of the effective prism, n is the index of refraction, and β the angle of incidence upon the prism surface. The expression given in Eq. (6) has been found useful in evaluating ϵ_P and ϵ_0 when D_{P_1} and D_{P_2} are known; it may also be regarded as giving the approximate magnitude of the major component of the average asymmetric distortion induced by prism effect.

The most striking difference to be noted in the two foregoing sets of equations is that ϵ_P is a variable dependent upon β while ϵ_T is invariant with respect to β. It is clear from a careful consideration of the above sets of equations, that the distortion, D_{P_1} produced by prism effect can be completely compensated by the distortion, D_T, produced by appropriate camera tipping for any selected value of β. For other values of β, the net distortion $D_P - D_T$ will be lower than D_P alone but not zero. Furthermore, even if complete neutralization of the asymmetric distortion is effected at the selected value of β, the value of ϵ_T will differ from ϵ_0. Consequently the displacement, $f\epsilon_T$, of the center image from the point of symmetry will differ from the displacement, $f\epsilon_0$, of the center image from the principal point. Therefore, while the net asymmetry of the distortion pattern produced by prism can be minimized by appropriate camera tipping, the values of $f\epsilon_0$ cannot be made equal to the value of $f\epsilon_T$.

In order to show more clearly the similarities and differences of the two types of asymmetric distortion, the succeeding sections deal with a specific example. For this purpose, the effect of a prism having a re-

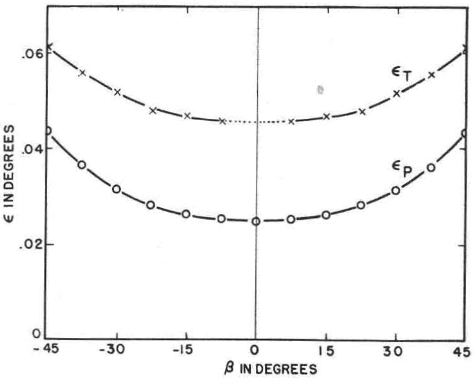

FIG. 1. Deviation, ϵ_p, in the principal plane produced by a prism having a refractive index of 1.5 and a prism angle of 3 minutes as a function of incident angle β. The upper curve shows the values of the angle of camera tipping, ϵ_T, necessary to produce the same amount of asymmetric distortion for each value of β.

fractive index, n, of 1.5 and a prism angle, α, of 3 minutes placed in front of a lens having a focal length of 150 mm., is considered. All computations in this example are for a principal plane of the prism.

2.1 COMPARISON OF ϵ_P AND ϵ_T

The lower curve, marked ϵ_P, in Figure 1 shows the variation of ϵ_P with β as computed from equation (7) for $\alpha=3$ minutes and $n=1.5$. It is clear that $\epsilon_P=\epsilon_0$ at $\beta=0°$, and increases steadily with increasing β, becoming almost double ϵ_0 at $\beta=45°$. It is evident that such behavior on the part of ϵ_P precludes any chance of completely neutralizing D_P at all values of β by an appropriate value of ϵ_T. However, complete neutralization can be achieved at any one selected value of β. Accordingly, the upper curve, marked ϵ_T, in Figure 1 shows the magnitude of the camera tipping, ϵ_T, that would just neutralize the asymmetric distortion caused by ϵ_P at several successive single values of β. Here, too, it is clear that ϵ_T is appreciably larger than ϵ_P.

2.2 LOCATION OF THE PRINCIPAL POINT AND THE POINT OF SYMMETRY

Values of D_{P_1}, D_{P_2}, $D_{P_1}-D_{P_2}/2$, and $f\epsilon_0$ have been determined with the aid of Eqs. (4), (5), and (6) for the case of the 3-minute prism under study. The separation of the principal point from the principal point of autocollimation is given by the quantity, $f\epsilon_0$, which does not change with β. This invariance of $f\epsilon_0$ with β is indicated by the straight line graph in Figure 2. If the quantity

$$\frac{D_{P_1}-D_{P_2}}{2} \text{ is set equal to } \frac{D_{T_1}-D_{T_2}}{2},$$

it is possible to compute by means of Eq. (3) a value of $f\epsilon_T$ for each value of β in the range from 7.5 to 45°. In this manner, the magnitude of the angle, ϵ_T, of camera tipping necessary to neutralize the asymmetric distortion arising from prism effect can be computed for each successive value of β. The quantity, $f\epsilon_T$, is the separation

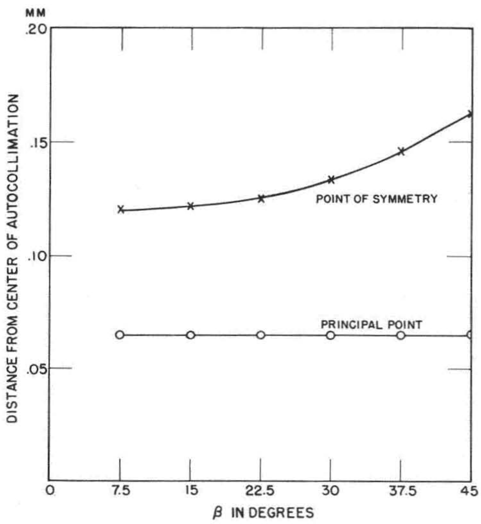

FIG. 2. The location of the point of symmetry and the principal point with respect to the center of autocollimation for the condition of a 3 minute prism of refractive index 1.5 placed in front of the lens.

of the point of symmetry from the principal point of autocollimation. This computation was performed and the results are shown in Figure 2, which clearly shows that a different location of the point of symmetry with respect to the center of autocollimation is obtained for each value of β. In addition, the values of $f\epsilon_T$ are in all cases greater than the value of $f\epsilon_0$.

2.3 COMPARISON OF ASYMMETRIC DISTORTION

The values of the asymmetric distortion produced by the 3-minute prism are shown, plotted with respect to β, in Figure 3 as the curve marked, D_P. The curve is shown for an entire diameter to emphasize the asymmetry of the distortion pattern. The curve, marked D_T shows the values of distortion with respect to β obtained by tipping the camera through an angle ϵ_T so chosen that $D_P = D_T$ for $\beta = 45°$. It is clear that the two curves are separated throughout the range for β greater than 0° but less than 45° with D_T greater than D_P. The actual difference is not great as is shown by the curve marked $D_T - D_P$ although the values of ϵ_T and ϵ_P are markedly different.

If one wishes, therefore, to neglect the differences in the location of various points

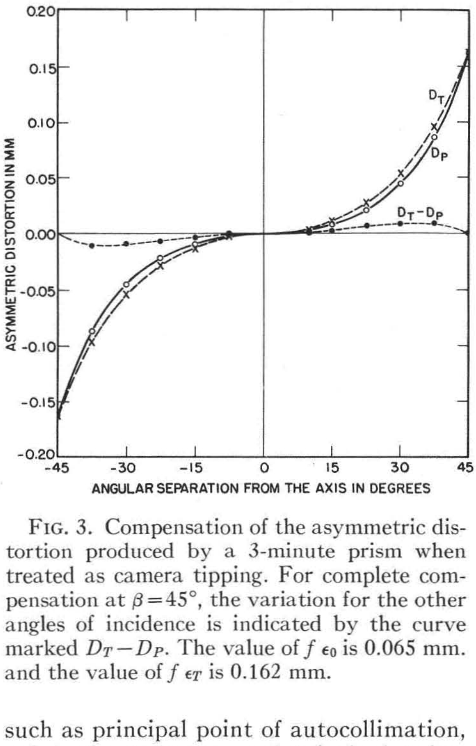

FIG. 3. Compensation of the asymmetric distortion produced by a 3-minute prism when treated as camera tipping. For complete compensation at $\beta = 45°$, the variation for the other angles of incidence is indicated by the curve marked $D_T - D_P$. The value of $f\ \epsilon_0$ is 0.065 mm. and the value of $f\ \epsilon_T$ is 0.162 mm.

such as principal point of autocollimation, point of symmetry, and principal point, then near neutralization of the asymmetric distortion can be achieved. This is illustrated for the case of the 3-minute prism more fully in Figure 4.

In Figure 4, curve 1 shows the values of the distortion along the diameter determined by the intersection of the principal plane of the effective prism and the focal plane, when the principal point of autocollimation is used as the zero point of coordinates; this is the point where the central image appears when the camera has been properly aligned for test. If the camera is tipped by an amount ϵ_T such that $f\epsilon_T = f\epsilon_0 = 0.065$ mm., then the central image falls on the principal point of the image plane.

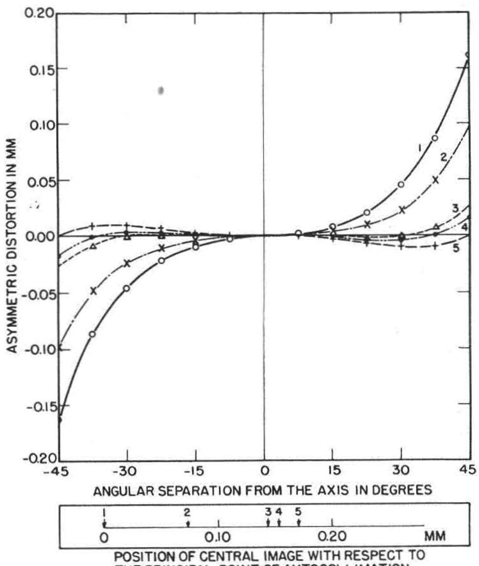

FIG. 4. Compensation of asymmetric distortion produced by a 3-minute prism placed in front of a lens ($f = 150$ mm.) by camera tipping. Curve 1 shows the initial distortion computed with respect to the principal point of autocollimation. Curve 2 shows the distortion when the camera is so tipped that the central image coincides with the principal point. Curves 3, 4, and 5 show the distortion when the prism effect is neutralized by camera tipping for successive values of $\beta = 30°$, 37.5°, and 45°. The bar graph at the bottom shows the successive positions of the central image.

Curve 2 shows the values of the distortion that will then be determined. For curve 2, the asymmetry of the distortion pattern is appreciably reduced from that shown in curve 1 but it is still quite pronounced.

Curves 3, 4, and 5 show the distortion values computed with respect to points of symmetry as successively determined for $\beta = 30°$; 37.5°; and 45° and for which the successive values of $f\epsilon_T$ are 0.135; 0.146; and 0.162 mm. The location of these various points with respect to the principal point of autocollimation is shown in the bar graph in the lower frame of Figure 4. For each of the last three cases, there are markedly lower values of the asymmetric distortion than shown in curves 1 and 2.

It is therefore clear that the asymmetry of the distortion pattern arising from prism effect can be appreciably reduced by appropriately tipping the camera. In the course of calibration, this would involve adjusting the camera so that the principal point of autocollimation was distant $f\epsilon_T$ from the center of collimation. However, while the radial component of the asymmetric distortion arising from prism effect can be partially neutralized by appropriate camera tip, the degree of compensation of the tangential distortion would be negligible.

3.0 Lens Tipping

When the subject of asymmetric distortion induced by camera tipping or prism effect is considered, one is likely to infer that a similar effect may be introduced by tipping of the entire lens with respect to the focal plane. Ideally the optical axis of a lens is a straight line, and in the ideal case the optical axis should be normal to the focal plane. In an actual lens, the optical axis is usually a broken line through the lens and cannot be located with precision. Usually a lens is seated in a camera cone in such a manner that an approximate optical axis is approximately perpendicular to the focal plane. It is self-evident that, if the lens is tipped with respect to the focal plane, or to state it differently, if the optical axis departs appreciably from the condition of normality with the focal plane, appreciable difference in definition will result. It is, however, not readily apparent what the effect on distortion may be, although as a first approximation it can be considered negligible.

This effect of lens tipping has been investigated recently at this laboratory. For this experiment a lens was carefully mounted in the precision lens testing camera so that its optical axis coincided as nearly as possible with the axis of the 0° collimator of the camera and its axis was moreover normal to the focal plane. A test negative was made. The lens was then turned approximately 0.6° counterclockwise about the vertical axis and a second negative made. The lens was then returned to its zero position and turned 0.6° clockwise about the vertical axis and a third negative made. The images at the various angles of incidence were examined and the negatives measured in the usual manner. The equivalent focal length and distortion were determined for each negative.

Analysis of the results showed the expected variations in resolving power. There were some variations in distortion but not large enough to be significant. On the basis of these measurements, it can be stated that the effect of tipping the lens by a small amount with respect to the focal plane is negligible insofar as asymmetric distortion is concerned. In any event, the asymmetric distortion that might be found arising from lens tipping will be less than 1/20 the amount produced by tipping the camera through a similar angle.

4.0 Tangential Distortion Produced by Prism

While it has been generally accepted that both tangential distortion and radial asymmetric distortion may be caused by a prism effect, there is some disagreement as to their relative magnitude. There is some tendency to believe that tangential distortion is the larger of the two (10). As this is not actually the case for prism effect, it seems worthwhile to present the results of a simple experiment that demonstrates conclusively the fact that the radial asymmetric distortion is the larger of the two.

The experimental arrangement consists of a copying camera, a square metal plate having a series of holes at approximately equal intervals along the two diagonals, and a 5.00 diopter prism. The metal plate was placed in the object plane, illuminated from the rear, and photographed at a reduction ratio of approximately 2:1. The appearance of the pattern of holes on the finished negative is shown in Figure 5. For the final experiment, two out of every three holes in the row designated III–IV were blanked out so that for this row only the central hole and those spaced at intervals 3, 6, 9, etc., would appear on the finished photograph. An initial exposure was then made of the array of holes. Then without disturbing the set-up in any way, a 5.00 diopter prism was placed in front of the lens with its axis parallel to row I and II and its base oriented toward row III. A second exposure on the same plate was then made and the finished plate processed to form a negative. The appearance of this final negative is shown in Figure 6.

To facilitate interpretation of the illustration, arrows and circles have been drawn thereon. The base of the arrows shows the location of the images formed by the first exposure, the tips of the arrows show the location of the displaced image formed by the second exposure. The circles are drawn with the undisplaced central image as center and pass through the un-

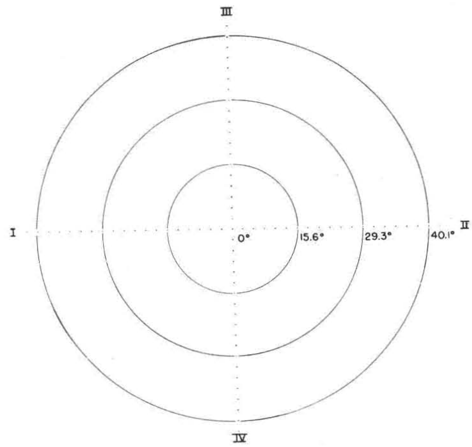

FIG. 5. Patterns of holes on target plate used to demonstrate radial asymmetric and tangential distortion. This shows the relative positions of the images prior to the introduction of prism.

displaced images located at equal distances from the center along all four radii. It is clear that the displacements of corresponding images are appreciably greater along the diameter, III–IV, coinciding with the principal plane of the prism than the transverse displacements of corresponding images along the diameter I–II. If a thread is stretched between corresponding points located on opposite sides of the center for the displaced row of images, it will be evident to the unaided eye, that this row of images is bowed.

Although this experiment was performed in an approximate manner, it is interesting to note the actual magnitudes of the various displacements. Table 1 shows the measured values of the distances, r_p, separating selected images along the radii III and IV from the deviated central image. It must be remembered that in the usual case of deviation by prism effect, values of r_p are the only distances measurable. The values of β listed in the table are determined from the spacings between holes in the target plate and the distance separating the lens from the plate. The reduction ratio was actually approximately 1.97 to 1; the focal length of the lens was 90.9 mm., and the power of the prism was 5.00 prism diopters. Values of the asymmetric radial distortion, D_r, were obtained from the values of r_p and the known values of r_m for the undeviated images which are not shown. Values of the deviation of the central image, $f \tan \epsilon_0$, were determined from D_r and β with the aid of Eq. (6). The tangent of ϵ_0 is substituted for ϵ_0 in Eq. (6)

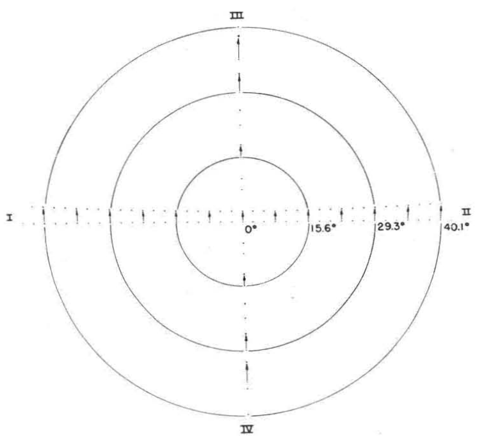

FIG. 6. Double exposure photograph showing appearance of the target pattern shown in Figure 5 before and after placing a prism in front of the lens. The base of the arrows show the initial undeviated image points; the tips of the arrows show the location of the same points after the 5.00 diopter prism was placed in front of the lens. Radial asymmetric distortion is evident along the III–IV diagonal. Pronounced bowing resulting from tangential distortion is clearly seen along the I–II diagonal.

TABLE 1

MEASURED VALUES OF THE DISTANCES, r_p, SEPARATING IMAGES ALONG RADII III AND IV FROM THE DEVIATED CENTRAL IMAGE

The separation of the deviated central image from the undeviated central image is 6.4 mm. Values of the asymmetric radial distortion and the computed values of $f \tan \epsilon_0$ are also shown.

Target No.	β	r_p	D_r	$f \tan \epsilon_0$
III	degrees	mm.	mm.	mm.
18	40.2	105.7		
15	35.1	89.7	−7.1	6.6
12	29.4	73.0	−4.3	6.6
9	22.9	55.7	−2.3	6.1
6	15.7	37.6	−1.0	6.7
3	8.0	19.1	−0.2	6.6
0	0	0.0	0.0	
3	8.0	19.6	0.2	
6	15.7	39.6	1.0	
9	22.9	60.3	2.3	
12	29.4	81.6	4.3	
15	35.1	104.0	7.1	
18	40.2			
IV				

because of $\epsilon_0 > 30$ minutes. The computed value of $f \tan \epsilon_0$ for the given prism is approximately 6.8 mm.; so the agreement is fairly good considering that the experiment was performed for illustrative purposes rather than for precise measurement.

Table 2 shows the magnitude of the displacement, t, of the deviated images from the undeviated images for the diameter I–II. The actual tangential distortion D_t is the difference in the value of t at a given value of β and the value of t for $\beta = 0$. It is noteworthy that the values of D_t are all positive and form a symmetric pattern about the central axis while the values of D_r are positive on one side of the zero point and negative on the other. The values of D_r and D_t are shown graphically in Figure 7 to show clearly the difference in character. It is clearly evident that in the case under consideration, the values of the radial asymmetric distortion are appreciably larger than the values of the tangential distortion. It must also be emphasized that in this specific case, both the displacement produced by radial and tangential distortions are all in the same direction as may be seen in Figure 6.

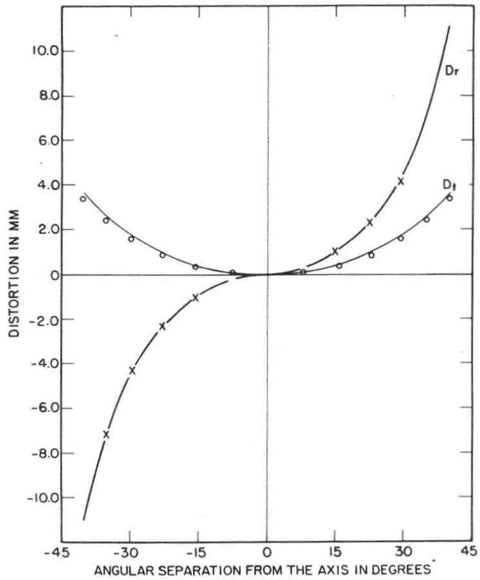

FIG. 7. Measured values of D_r and D_t for the negative shown in Figure 6. Values of V_r are indicated by X's; values of D_t are shown by circles. The solid curves show the theoretical values of the same quantities.

4.1 COMPUTATION OF TANGENTIAL DISTORTION PRODUCED BY PRISM

In the field of spectroscopy, the existence of curvature or bowing of spectrum lines has long been known and has been properly attributed to variation of the deviation of light by the prism for rays, not contained in a principal section. Equations giving the radius of curvature of the spectrum lines are given in various textbooks (19). Tangential distortion arising from prism is produced in the same manner as the bowing of spectrum lines. The chief differences are that the angle of the prism is much smaller in aerial camera problems and the angle of departure from the principal section is much greater. While it is possible to evaluate the probable magnitude of the tangential distortion in a theoretical manner, it seems simpler to make a direct measurement of the deviation. Accordingly, the 5.00 diopter prism was placed on the optical bench, and a measurement was made of the deviation, ϵ_t, in the direction parallel to the principal section of the prism of rays lying in a plane normal to the principal section and the surface of the prism and incident on the prism surface at various inclinations, β. The results of measurement of ϵ_t are shown in Table 3, which also shows the ratios ϵ_t/ϵ_0, ϵ_r/ϵ_0, ϵ_r/ϵ_t and the values of sec β. The values of ϵ_r/ϵ_0 were determined with the aid of Eq. (7), where $\epsilon_r = \epsilon_P$; in addition a few spot checks of the reliability

TABLE 2

MEASURED VALUES OF THE TRANSVERSE DISPLACEMENT t OF RELATED IMAGE POINTS DEVIATED BY PRISM FROM THE UNDEVIATED POSITION

Values of the tangential distortion, D_t, are also shown.

Target No.	β	t	D_t
I	degrees	mm.	mm.
18	40.1	9.74	3.34
15	35.1	8.77	2.37
12	29.3	7.90	1.50
9	22.8	7.22	0.82
6	15.6	6.75	.35
3	7.9	6.48	.08
0	0	6.40	.00
3	7.9	6.48	.08
6	15.6	6.75	.35
9	22.8	7.24	.84
12	29.3	7.95	1.55
15	35.1	8.80	2.40
18	40.1	9.80	3.40
II			

TABLE 3

MEASURED VALUE OF THE DEVIATION, ϵ_t, IN A DIRECTION PARALLEL TO THE PRINCIPAL SECTION OF A 5.00 DIOPTER PRISM AS A FUNCTION OF THE INCIDENT ANGLE β OF RAYS IN A PLANE NORMAL TO THE PRISM SURFACE AND TO THE PRINCIPAL SECTION

Values of ϵ_t/ϵ_0, ϵ_r/ϵ_0, and sec β are also shown. Values of $\Delta=\epsilon_r/\epsilon_t-\sec\beta$ are also listed.

β	ϵ_t	ϵ_t/ϵ_0	ϵ_r/ϵ_0	ϵ_r/ϵ_t	sec β	Δ
degrees	minutes					
0	18.60	1.0000	1.0000	1.0000	1.0000	0.0000
5	18.68	1.0043	1.0064	1.0021	1.0038	−.0017
10	18.82	1.0118	1.0256	1.0136	1.0154	−.0018
15	19.05	1.0242	1.0592	1.0342	1.0353	−.0011
20	19.38	1.0419	1.1088	1.0642	1.0642	.0000
25	19.82	1.0656	1.1760	1.1036	1.0134	.0002
30	20.38	1.0957	1.2660	1.1554	1.1547	.0007
35	21.05	1.1317	1.3836	1.2226	1.2208	.0018
40	21.90	1.1774	1.5384	1.3066	1.3054	.0012
45	22.84	1.2279	1.7412	1.4180	1.4142	.0038

of the equation were also made by actual measurement.

The relative magnitudes of ϵ_r and ϵ_t are shown graphically in Figure 8, where the ratios of ϵ_r/ϵ_0 and ϵ_t/ϵ_0 are plotted. It is of interest to note that the following approximate relation exists between ϵ_r and ϵ_t.

$$e_r = e_t \sec\beta$$

or

$$e_t = e_r \cos\beta \tag{8}$$

where ϵ_r is the deviation of a ray lying in the principal section and inclined at angle β to the surface of a thin prism, and ϵ_t is the deviation in a plane parallel to the principal section of a ray lying in a plane normal to the principal section and the prism surface and inclined at angle β to the surface of the thin prism. It is realized that Eq. (8) is only an approximate relation but it is of interest to note from a comparison of ϵ_r/ϵ_t and sec β as shown in Table 3, that the maximum departure from equality does not exceed $\pm 0.2\%$ for a 5.00 diopter prism. For the purpose of determining the magnitude of tangential distortion in aerial cameras, where the effective prism is likely to be small, Eq. (8) is sufficiently accurate.

The actual computation of the radial distortion is fairly simple for the plane of the principal section. It is given by the relation

$$D_r = \frac{f\tan\epsilon_r}{\cos^2\beta} - f\tan\epsilon_0 \tag{9}$$

For small values of ϵ, Eq. (9) becomes

$$D_r = f\left(\frac{\epsilon_P}{\cos^2\beta} - \epsilon_0\right) \tag{10}$$

which is identical with Eq. (6) when ϵ_P is substituted for ϵ_r. Values of the tangential distortion for the plane normal to the principal section are given by the relation

$$D_t = \frac{f\tan\epsilon_r}{\cos\beta} - f\tan\epsilon_0 \tag{11}$$

which for small ϵ is equivalent to the expression

$$D_t = f(\epsilon_r - \epsilon_0) \tag{12}$$

or

$$D_t = f(\epsilon_P - \epsilon_0) \tag{13}$$

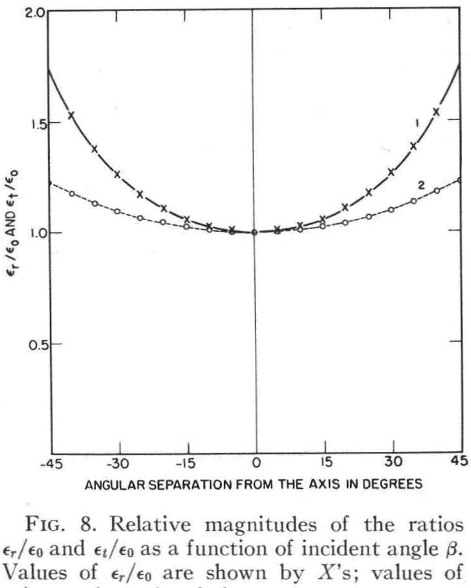

FIG. 8. Relative magnitudes of the ratios ϵ_r/ϵ_0 and ϵ_t/ϵ_0 as a function of incident angle β. Values of ϵ_r/ϵ_0 are shown by X's; values of ϵ_t/ϵ_0 are shown by circles.

4.2 EVALUATION OF D_t WHEN D_r IS KNOWN

It is clear from the information contained in the foregoing section that a fairly close relationship exists between radial asymmetric and tangential distortion produced by prism. Moreover, when the radial asymmetric distortion takes place in the principal section, it is much larger than the tangential distortion for the meridian at right angles to the plane of the principal section. It has been demonstrated that the magnitude of ϵ_P and ϵ_0 can be readily evaluated from analysis of the radial asymmetric distortion. Accordingly with f, ϵ, and ϵ_0 known, the values of D_t can be determined sufficiently accurately with the aid of Eq. (13).

The problems that arise when the principal sections of the prism do not coincide with the test meridian are somewhat more complex. To show what actually happens another double exposure negative was made in accordance with the procedure used in producing Figure 6, and a reproduction of this second negative is shown in Figure 9. In making this negative, the principal section of prism interesects the image plane along the meridian line inclined at 45° to diameters I–II and III–IV. The base of the prism was oriented to produce displacement of the central image as shown in the figure. For this experiment, alternate holes in the target plate were covered. The arrows and circles are drawn in the same manner as described for Figure 6. The figure is self-explanatory. It is clear that both radial and tangential distortion are present along both rows.

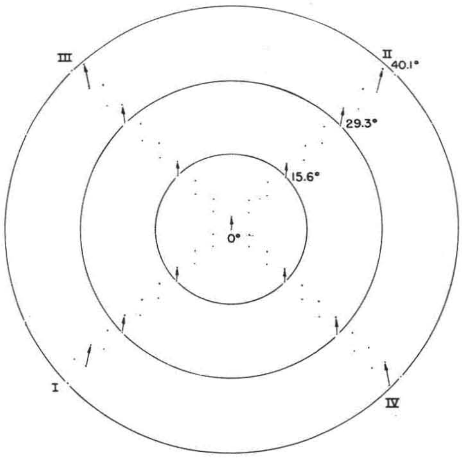

FIG. 9. Double exposure negative showing appearance of the target pattern shown in figure 5 before and after placing the 5.00 diopter prism in front of the lens. In this instance, the principal section of the prism intersects the image plane along the meridian line inclined at 45° to diameters I–II and III-IV. Radial asymmetric and tangential distortions are now evident along both diameters.

While the actual results of measurement are not here presented, it is found that

$$D_{t_0}{}^2 = D_{t_1}{}^2 + D_{t_2}{}^2 \tag{14}$$

and

$$D_{r_0}{}^2 = D_{r_1}^2 + D_{r_2}{}^2. \tag{15}$$

Where the subscript, 0, refers to values obtained under the conditions used in producing Figure 6 and the subscripts, 1 and 2, refer to the values of the distortions obtained under the conditions used in producing Figure 9.

While the variation of D_r and D_t with azimuth has not been treated theoretically in this discussion, it should be mentioned that both quantities vary with azimuthal angle. D_r has its maximum value in the principal plane of the prism and has zero value in the plane at right angles; it is a vector quantity and may be resolved into components about any set of orthogonal axes desired in accordance with the usual procedure. This property is useful when the principal plane of an unknown prism does not lie in one of the coordinate axes formed by the collimator banks of the calibrator. The prism effect is then found independently for each diagonal and the resultant values projected upon the rectangular coordinate system formed by the lines joining opposite pairs of collimation markers. D_t has its minimum value in the principal plane of the prism and its maximum in the direction at right angles to the principal plane; it is also a vector quantity and can be resolved into components in the same manner as D_r.

If preferred, the maximum value of D_r can be obtained with the aid of Eq. (15); the azimuthal angle for maximum D_r is readily determined from the values of D_{r_1} and D_{r_2}. The maximum value of D_t can then be computed as shown in the discussion following Eq. (13).

4.3 EFFECT OF TANGENTIAL DISTORTION ON CHOICE OF PRINCIPAL POINT

A great deal has been written on the subject of what is the best point to regard as the principal point of the camera for use by the photogrammetrist (2, 10, 11, 12, 13, 14, 15, 16). At present, there is no

TABLE 4

COMPARISON OF RELATIVE MAGNITUDES OF RADIAL ASYMMETRIC DISTORTION D_r, TANGENTIAL DISTORTION D_t, AND SEPARATION OF THE PRINCIPAL POINT FROM THE CENTER IMAGE, $f \tan \epsilon_0$, FOR A LENS HAVING A FOCAL LENGTH OF 150 MM. AFFECTED BY A THIN PRISM ($\alpha=3$ MINUTES) OF REFRACTIVE INDEX 1.5, $f \tan \epsilon_0=0.065$ MM.

	D_r	D_t	$\frac{D_r}{D_t}$	$\frac{D_r}{f \tan \epsilon_0}$	$\frac{D_t}{f \tan \epsilon_0}$
degrees					
0	0.000	0.000		0.00	0.00
7.5	.002	.001	2.00	.03	.02
15	.009	.004	2.25	.14	.06
22.5	.021	.009	2.33	.32	.14
30	.045	.017	2.65	.69	.26
37.5	.086	.030	2.87	1.32	.46
45	.162	.048	3.38	2.49	.74

universal agreement concerning which of the three points, principal point, principal point of autocollimation, and point of symmetry, is to be preferred. Recently, it was suggested that a move be made in the direction of the point of symmetry and that it be renamed the calibrated principal point (17). Such a procedure is perhaps justified in the absence of prism effect.

However, the introduction of an apparent camera tip to compensate for asymmetric distortion produced by prism effect is not perhaps the best answer. In comparatively recent years, a new type of distortion usually referred to as tangential distortion D_t (18) has been reported. This is a distortion which produces a displacement of a point in the direction at right angles to the radius drawn from the center of the picture. It has been shown in this section that a distortion of this type can be deduced from the prism effect. Values of the tangential distortion, D_t, produced by prism can be computed with the aid of Eq. (13).

Relative magnitudes of the radial asymmetric distortion D_r and the tangential distortion D_t are shown in Table 4 for a lens of 150 mm. focal-length, affected by a prism of index 1.5 and angle $\alpha=3.0$ minutes. The distortions D_r and D_t are measured in meridians at right angles to each other and maintain a constant relation to each other for a given orthogonal pair of meridians. Consequently, if the value of the shift of the center image from the principal point, $f\epsilon_0$, is computed from a given set of values of the radial asymmetric distortion, it is possible to compute the magnitude of the tangential distortion from the value of $f\epsilon_0$. The last column in the table shows the ratio of $D_t/f\epsilon_0$ for each value of β.

This type of distortion cannot be compensated by small camera tipping and is best eliminated by rejecting the camera that shows an excessive amount of tangential distortion. Some tentative tolerances have already been considered ranging from 0.015 to 0.030 mm. total. Table 4 shows the relation between tangential distortion and $f\epsilon_0$ for several values of β. From these considerations, it is clear that a tolerance placed on D_t automatically places a limit on the magnitude of the prism effect. This in turn reduces the asymmetric distortion to be expected, and reduces the separation of the principal point and point of symmetry for any lens-camera combination that is acceptable for use.

Recently improvements in lens centering procedures have reduced the amount of prism effect and the attendant tangential distortion. The effect is still present in measurable quantities although to a lesser degree in cameras of recent manufacture. It is of interest to consider the range of values of $f\epsilon_0$ for a number of cameras calibrated during the last few years. Table 5 shows the number of cameras for which the measured value of $f\epsilon_0$ falls within certain narrow specified ranges for the three focal lengths most commonly encountered. The probable deviation ϵ_0 produced by the prism effect can be inferred from Table 6 which shows the values of $f\epsilon_0$ as a function of ϵ_0. The prob-

TABLE 5

SEPARATION OF THE CENTER IMAGE FROM THE PRINCIPAL POINT. DISTRIBUTION OF VALUES OF $f \tan \epsilon_0$ FOR 113 CAMERAS OF THREE FOCAL LENGTHS

Range of $f \tan \epsilon_0$	Number in Range Having a Focal Length of		
	130 mm.	150 mm.	210 mm.
mm.			
0.000–0.010	0	35	0
.011– .020	4	33	0
.021– .030	3	7	3
.031– .040	1	5	4
.041– .050	0	4	2
.051– .060	0	0	1
.061– .080	0	2	3
.081– .100	0	1	2
.101– .120	0	0	1
.121– .160	0	0	2
over .161	0	0	0
Total	8	87	18

able value of the tangential distortion, D_t, can be inferred from Table 7 which shows values of $f\epsilon_0$ corresponding to specified values of D_t.

It can be seen from Table 5 that 68 out of the 87 cameras of 6 inch focal-length, or 78 per cent, have a value of $f\epsilon_0$, that is less than 0.020 mm. Consequently, for these 68 cameras, the tangential distortion should not exceed 0.015 mm., which has been tentatively suggested as a tolerance for this aberration. Similarly, 10 out of the 18 cameras of $8\frac{1}{4}$ inch focal-length have values of $f\epsilon_0$ less than 0.060 mm. which

TABLE 6

VALUES OF $f \tan \epsilon_0$ AS A FUNCTION OF f AND ϵ_0

f	130 mm.	150 mm.	210 mm.
ϵ_0	$f \tan \epsilon_0$		
minutes	mm.	mm.	mm.
0.25	0.010	0.011	0.015
.50	.019	.022	.030
.75	.028	.033	.045
1.00	.038	.044	.061
1.50	.057	.066	.091
2.00	.075	.087	.122
2.50	.094	.109	.152
3.00	.113	.130	.183

TABLE 7

VALUES OF $f \tan \epsilon_0$ NECESSARY TO PRODUCE A GIVEN VALUE OF THE TANGENTIAL DISTORTION D_t FOR SEVERAL VALUES OF β

β	30°	37.5°	45°
D_t	$f \tan \epsilon_0$		
mm.	mm.	mm.	mm.
0.015	0.058	0.033	0.020
.030	.115	.066	.040
.045	.173	.099	.060

corresponds to a tangential distortion of 0.015 mm. for a total field of 60°.

If the tolerance on tangential distortion be set as high as 0.030 mm. 7 out of 87 cameras having a 6 inch focal-length would be outside of tolerance, and 2 out of the 18 cameras of $8\frac{1}{4}$ inch focal-length would have tangential distortion in excess of the tolerances. For this tolerance of $D_t = 0.030$, the radial asymmetric distortion for $\beta = 45°$ would amount to 0.100 mm., and the separation of the center image and point of symmetry for $\beta = 45°$ would be 0.100 mm., while the separation of center image and principal point would be 0.040 mm. In view of the magnitude of D_r for $D_t = 0.030$ mm., it seems that a value of $D_t = 0.015$ mm. may be preferable. For $D_t = 0.015$ mm., the separation of the point of symmetry from the center image is equal to or less than 0.050 mm.; the separation of the principal point from the center image is equal to or less than 0.020 mm., and the separation of the principal point from the point of symmetry is equal to or less than 0.030 mm. Consequently, whether the point of symmetry or principal point is set at the center of collimation, the other will be not more than 0.030 mm. distant, and the maximum radial asymmetric distortion will not exceed 0.030 mm. for any point selected as center between the point of symmetry and the principal point.

5.0 DISCUSSION

This is the final paper is a series dealing with problems associated with camera calibration. This series has dealt with various sources of error that are likely to be present and their effect on the accuracy of the calibration. Considerable emphasis has been placed on prism effect throughout this series.

It should be emphasized that it is not

the author's contention that all forms of asymmetric distortion can be explained in full by treating the actual lens as a combination of an ideal lens and a thin prism. It is realized that in general a simple prism cannot account in full for all the deviations noted. However, for all of the cameras calibrated at the National Bureau of Standards during the past 20 years, there has not been a single instance where the asymmetric part of the observed distortion patterns could not be in large measure accounted for by ascribing it to a simple thin prism. It is true that there generally remains small residual distortion that is not fully accounted for; these residuals are however usually negligibly small in comparison to the initial values of the asymmetric distortion. It seems, therefore, that until a more elegant and theoretically sounder analysis of the problem is forthcoming, that the relatively simple explanation in terms of prism effect provides a sufficiently close approximation upon which to base computations used in the course of camera calibration.

6.0 Acknowledgments

The author expresses his appreciation to other members of the staff for assistance rendered during the course of this work. The negatives used in the confirmatory experiments were made by Mr. W. P. Tayman. Measurements on the negatives were made by Mr. W. R. Darling. The drawings were made by Mr. E. C. Watts.

7.0 References

1. Gardner, I. C. and Case, F. A., "Precision Camera for Testing Lenses." *J. Research NBS*, 18, 449 (1937) RP984.
2. Washer, F. E. and Case, F. A., "Calibration of Precision Airplane Mapping Camera." Photogrammetric Engineering, XVI, 502 (1950); *J. Research NBS*, 45, 1 (1950) RP2108.
3. Gardner, I. C., "Relation of Camera Error to Photogrammetric Mapping. *J. Research NBS*, 22, 209 (1939) RP1177.
4. Washer, F. E. "Locating the Principal Point of Precision Airplane Mapping Cameras." *J. Research NBS*, 27, 405 (1941) RP1428.
5. Washer, F. E., "Sources of Error in Camera Calibration." Photogrammetric Engineering, XX, 500 (1954).
6. Washer, F. E., "Sources of Error in Various Methods of Camera Calibration." Photogrammetric Engineering, XXII, p. 727 (1956)
7. Washer, F. E., "The Effect of Camera Tipping on the Location of the Principal Point." *J. Research NBS*, 57, 31 (1956) RP2691.
8. Washer, F. E., "A Simplified Method of Locating the Point of Symmetry. Photogrammetric Engineering, XXIII, p. 75, (1957).
9. Washer, F. E., "The Effect of Prism on the Location of the Principal Point." Photogrammetric Engineering, XXIII, p. 520, (1957).
10. Sharp, J. V. and Hayes, H. H., "Effect on Map Production of Distortions in Photogrammetric Systems." Photogrammetric Engineering, XV, 159, (1949).
11. Sewell, E. D., "Field Calibration of Aerial Mapping Cameras." Photogrammetric Engineering XIV, 363 (1948); Manual of Photogrammetry by the American Society of Photogrammetry, published 1952, pp. 137 to 176.
12. Field, R. H. "A Device for Locating the Principal Point Markers of Air Cameras." *The Canadian Surveyor*, July 1949 (N.R.C. No. 1979).
13. Carman, P. D., "Photogrammetric Errors from Camera Lens Decentering." *J. Opt. Soc. Am.*, 29, 951 (1949).
14. Roos, W., "Über die Definition des Bildhauptpunktes und der Aufnahmeachse." *Allgem. Vermess Nachr.* (Bildm. u. Luftbildwesen) p. 235 (1950).
15. Merritt, E. L., "Methods of Field Camera Calibration." Photogrammetric Engineering, XVII, 610 (1951).
16. Rock, D. L., "Field Determination of the Center Cross. Photogrammetric Engineering, XVII, 596 (1951).
17. Roelofs, R., "Distortion, Principal Point, Point of Symmetry and Calibrated Principal Point." *Photogrammetria*, II, 49 (1950–1951).
18. Pennington, J. T., "Tangential Distortion and its Effect on Photogrammetric Extension of Control, Photogrammetric Engineering, XIII, 135 (1947).
19. Martin, L. C., Geometrical Optics. P. 104, Published 1956 by Philosophical Lib., Inc.
20. Emsley, H. H. and Swaine, W., "Ophthalmic Lenses. P. 80. Sixth Edition Published 1951 by Hatton Press, Ltd.
21. Duke-Elder, S., "The Practice of Refraction." P. 310. Published 1938 by P. Blakistons Son & Co.
22. Martin, L. C., "Technical Optics." Vol. 1, p. 326. Published 1948 by Pitman Publishing Corporation.