ARRAY ALGEBRA AUTOMATION OF 4-D IMAGING AND RANGE SENSING

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ABSTRACT

Mathematical photogrammetry and geodesy pioneered some early applications of matrix and tensor calculus in the general theory of estimation and statistics. This theory together with fast transforms of signal processing was expanded since 1968 in author's inventions of array algebra and loop inverses in Finland and Sweden, leading to 1975 employment with Duane Brown and other US pioneers of this field. Some 1975-83 R&D projects developed the fast finite element network adjustment for automated terrain edit, compression and progressive sampling of digital correlation and Least Squares Matching (LSM). This combined with fast strip and block adjustment techniques started some experimental 1984 array algebra work with GDE/BAE Systems and Helava Associates in San Diego during the softcopy transition period from analytical to digital. Automated edit of image, terrain and feature LSM registrations was integrated into several data-to-info conversion and fusion prototypes. Their theory and algorithms together with Least Squares Compression (LSC) predecessor of wavelet theory were reported in past ASPRS/ISPRS, SIAM and other conventions. This work resulted in nonlinear array algebra expansion of matrix, tensor, differential and integral calculus by late 1980's. Its latest direct nonlinear solution technology of Fast Taylor Transform (FTT) tensor decomposing is enabling advanced 4-D imaging and range sensing concepts. The automated edit and fusion of these compressive registration and bundle adjustment techniques in array algebra are ideally completed in real-time as an integral part of future sensor design. This challenge requires various communities to adopt and continue the reported work.

BACKGROUND AND HISTORICAL NOTES

The early multi-linear tensor decomposing and other fast signal processing operators of array algebra were found in my 1972-80 studies of terrain data edit and compression, including the fast finite element network solution. It provided the foundation for automation of the 1976-77 image and terrain registration inventions of Least Squares Matching (LSM). The 1978-80 automated terrain edit and compression studies with Dr. R. Helmering of DMAAC as the contract monitor also found the array algebra FFT, fast Karhunen-Loeve and fast cosine transforms enabling the 1985 decompression board design of an early softcopy WS technology in San Diego.

This paper focuses on my recent work in nonlinear differential and integral array algebra of my private studies since 2004 retirement from BAE Systems in San Diego. This work in the foundations of math and engineering sciences started in 1980's and was reported at 1990/1992 ASPRS, 1992 ISPRS and at 1995, 1999 SIAM convention papers. It further expands and unifies the matrix and tensor calculus of the early multi-linear array algebra. These new notations of array calculus express the differential and integral operators of nonlinear loop inverse estimation in a more intuitive way than the tensor notations of Blaha's Q-surface operators of Taylor-series. They allow interpreting Blaha's tensor approach of least squares estimation in terms of the nonlinear loop inverse expansion of sequential or Kalman updating with a nonlinear condition adjustment and wavelet theory. They also removed the rank restriction of Schnabel's nonlinear tensor method of the SIAM community and resolved the challenging problem of nonlinear tensor or array decomposing of Integral Least Squares Matching (ILSM).

It was, and still is, difficult to get the papers of nonlinear array algebra accepted for a peer review and publication. My 1990-92 convention papers introduced some crude word processing notations for the exponential vector contractions with matrix and tensor derivatives of Taylor series. The notations were refined using commercial equation editor software (with a translator to LaTex) for the 1995 and 1999 SIAM papers. Dr. Georges Blaha, my former colleague and expert of Hotine's tensor calculus in mathematical geodesy, could catch some mistakes in my manuscripts and I kept finding improvements whenever I had some time available for this publication effort since mid 1990's. Two technical Journals (including PERS) rejected the mere idea of starting their peer review process as the paper was "too mathematical" for their reviewers and readers.

The new theory was conceived by incremental trial and error experiments in improving some early LSM prototypes of GDE/BAE Systems IRAD efforts that already had won the internal and external (presented in 1993 SPIE convention) competitive tests with the commercial standard (HRC) and my experimental least squares expansion of cross correlation in terrain extraction. By slight modifications of existing LSM algorithms, pieces of the new theory could be quickly implemented showing improvements in the pull-in range, rate of convergence and overall quality – with a superior speed to HRC. This leading edge nonlinear theory appeared to be valid but how to sell it for a production oriented development if nobody can verify it?

I was laid off twice during the early invention period of nonlinear array algebra but was recalled to do some hands-on systems prototyping of LSM sales efforts until the 2004 lay-off and early retirement. Dr. Blaha working in Florida part-time for Brown's family owned business in industrial close-range photogrammetry had similar problems to fund this basic research of nonlinear tensor expansion of Taylor-series in the geometric adjustment theory of Q-surface. He helped in a pre-review of my SIAM paper for its publication submittal about the Q-surface interpretation in terms of nonlinear array algebra and loop inverses. After months of telephone and E-Mail exchanges we met in Florida and worked out until 4-5 a.m. of the last night before we could conclude that the starting point of my extended tensor notations and nonlinear Lm-inverse concept was correct. He died in 2008 of cancer (like Hotine, Brown and other pioneers in this field before him) without getting his latest work published since the 1991 submittal of his Q-surface tensor theory of (Blaha, 1994).

After joining the SIAM community I kept submitting modified 1999 convention versions of my paper to its journals until the SIMAX editor found one reviewer who admitted of not fully understanding the details of the paper although "it looks like Herculean effort in the foundations of applied mathematics – but cannot be published as no regular SIAM reader would understand it". I followed the advice of splitting the paper into two more digestible parts and leaving out difficult details of the proofs and derivations - and got it published in 2002! This did not help to sell the prototypes of my IRAD work for large-scale production efforts so my team members were laid-off or taken to other projects. In 2004 I followed the past examples of Duane Brown and Uki Helava (and many unemployed or early retired scientists today) to continue my work as a consultant and business owner. The goal was the multi-dimensional expansion of FTT solutions of ILSM and LSC theories and refinement of my "hyper sensor" design ideas since 1975 with this new estimation theory of nonlinear matrix, tensor, integral and differential calculus.

Since 2004, I made three ASPRS presentations and submitted seven proposals in response to the yearly requests of various agencies to small businesses on solving some specially targeted problems. Many of these problems were already resolved in my past array algebra R&D or were ideal candidates to fund the latest nonlinear array algebra expansion. All of my proposals were rejected. No interest or feedback was shown to my SIMAX and ASPRS papers - although recent publications and convention papers claimed inventions of "new" techniques unaware of my published or reported work at various conventions since 1972. This experience appears universal despite the modern day communications and electronic libraries of many overlapping communities of applied math and engineering disciplines. This ASPRS farewell paper to my active R&D career (other than small lecture and consultation activity) is the final attempt to stir the R&D of this fertile field and is organized as follows.

A compressive Data-To-Info (DTI) conversion system is introduced using Fast Taylor Transforms (FTT), Integral Least Squares Matching (ILSM) and Least Squares Compression (LSC) with the geometric fusion technology of Epipolar Bundle Adjustment (EBA). The theory of LSC of my 2006 ASPRS Fall Convention paper illustrates the basic idea of loop inverse estimation in terms of linear condition adjustment – a predecessor of wavelet theory dating to the surveying math of ancient times. "Loop inverses" expand the theories of general matrix inverses, signal processing, applied mathematics and unbiased estimators of math statistics. An example of the loop inverse estimation theory of relativity in free net adjustment is mentioned using a scale invariant triangle adjustment of range measurements. The nonlinear theory of loop inverse estimation and its 1-D FTT solution idea of ILSM are outlined. It provides the foundations of nonlinear differential and integral calculus for the advanced on-board DTI application of automated 4-D imaging and range sensing.

COMPRESSIVE FUSION OF AUTOMATED DATA-TO-INFO CONVERSION

Nonlinear array integral and differential calculus of Fast Taylor Transforms (FTT) has the unprecedented potential to reduce the number of ILSM arithmetic operations by 90-99% from the proven discrete LSM prototypes. This saving is achieved using advanced paper and pencil derivations beyond the known capabilities of computers and programmers today. Several, say n-1, nearly epipolar slave images can then be registered to one reference image (or in suitable slave-to-slave pairings) using the closed-form inverse FTT array (scalar, vector, matrix or tensor) solution of ILSM, ideally in a parallel and hardwired pipeline of on-board data-to-info (DTI) conversion processes.

The parameters of reference-to-slave ILSM image registrations consist of 1) a few relative orientation parameters of y-shifts per slave image, 2) one adjusted x-shift parameter per match point at grid locations of the reference image and 3) one radiometric bias correction per match point per slave image. The automated validation of each image pair registration is made feasible by the feedback loop of the pull-in values where the (properly scaled) x-shift output in the early and easily registered parts of pipeline feeds the point variant component of initial values in subsequent (otherwise increasingly more difficult) registrations of ILSM. The resulting multi-image ILSM registration model thereby converts the gray value data of n-1 slave images into highly compressive geometric and radiometric scene information using the fusion technique of Epipolar Bundle Adjustment (EBA) in one n-ray model of a "hyper sensor" image sequence.

A starting 1975 idea (discussed with Duane Brown and in my convention, proposal and white papers) of the hyper sensor design was an aerial m-cone frame camera with m=4-9 where the view angles of the cones or rigidly attached narrow-angle sub-cameras form a 2x2 or 3x3 array to cover one wide angle view. The sub images of one "hyper exposure" require no point variant registrations of their nearly zero vs. 80-100% overlap areas of the discussed general case of multi-view imaging technology. An on-board DTI registers and compresses the wide-angle "hyper images" of multiple views among n=3-7 consecutive "hyper exposure" stations of a long flight sequence. The fusion of the wide-angle hyper images is trivial in the special case of virtually the same view angle of zero base-to-height (b/h) geometry. This fusion can be applied (when needed) for multi-view imaging at a ground station after the on-board ILSM, EBA and LSC compression, transmittal, decompression and absolute or "datum fusion" bundle adjustment. The 4-D multi-ray and multi-spectral fusion of all r overlapping images of a given object space area provides an image enhancement or averaging with sqrt (r) times better resolution.

The EBA edited and fused shifts of the highly redundant ILSM observed values are already compact before their on-board LSC compression is started. The y-shift model is point invariant taking only a few words of encoding per slave image. The smoothly behaving adjusted x-shift per one match point is common for n-1 slave images at a grid spacing of about p= 2-5x2-5=4-25 pixels. EBA therefore makes it possible to resample the gray values of n-1 slave images from the reference image using p(n-1) = 8-150 times fewer adjusted x-shifts that correspond to the elevation variations of the object space in the 4-D epipolar coordinate system of object space. The ILSM/EBA processes of DTI conversion therefore act as pre-compressors for LSC of the fused terrain and feature heights.

The adjusted terrain and feature elevation info of EBA is more compact (even with errorless compression) than the compressed gray values of the reference image. Encoding the registration info of the shift parameters of slave images gets negligible in comparison to the gray value encoding of the reference image of one n-ray stereo sequence. The remaining task compresses the differential radiometric bias info of the slave images. This info is needed in the slave image reconstructions for gray value corrections after re-sampling the slave images using the decompressed reference image and shift parameters. This technique allows interpolation of fictitious slave images at any imaging time within the sequence.

The additive bias correction per match point per slave image is close to the average gray value difference of the slave match window at the shifted location from the constant 2-D Taylor term of reference window. It is caused by systematic scene illumination differences due to the changed sensor view or by the scene changes (feature movement or occlusions) between the imaging epochs. Choosing the reference image from the middle portion of an image sequence minimizes the overall bias corrections. Since the EBA adjusted x-shift info is so compact, the sequence can be split into smaller pieces, each with a temporary reference image registered to the most central reference image. Only one reference image for the entire sequence needs encoding together with the negligible additional encoding of the reference-to-reference registration info.

The number of bias terms in a slave image is 4-25 times less than the number of its gray values. The magnitude of low-pass bias terms at, say, 90-99% majority of points is about, say, 10-100 times smaller than the original slave gray values in typical scenes. The n-1 consecutive bias images of a sequence are highly correlated along the time variable allowing high compression in the fashion of MPEG techniques. The remaining 1-10% bias corrections are valuable for the image analyst as they provide the minute change or movement detection, occlusion and other clues about the scene contents. Their average is close to the constant Taylor terms in the slave and reference windows but because they are the exception, their encoding requirement is reduced by the said factor of 10-100.

Thus, encoding the n-image sequence consists of the converted ILSM registration info of n-1 slave images and by a temporary on-board encoding/decoding buffering of the reference image until the reference image of the next sequence gets available, say, in 3n-ray hyper stereo sensor. The reference image of the master sequence has a large base-to-height ratio (view change) with the reference images of the slave n-ray sequences. The completed inter-sequence registrations provide the pull-in values for these extra ILSM processes. They also feed the absolute strip and block adjustment techniques of array algebra at the ground station for the datum fusion of dissimilar and multi-temporal info with the "hyper tie point" technique of Entity or terrain shape LSM, (Rauhala, 1996).

In summary, the data-to-info conversion of ILSM reduces LSC encoding of 3n-ray (say, 27-ray) stereo images from same flight path to an equivalent of encoding a traditional 2-ray stereo pair. It enables the welcome evolution at the ground station for automated analysis processes of these pre-registered and triangulated images to exploit the compressed info. The bottleneck of analyst dependent validation of these automated processes is removed using the bias correction clues and instant 4-D (x,y,z,t) location info of found objects with 0.05-0.2 pixels accuracy of observed parallaxes. The high redundancy of metric 4-D (vs. regular visual video) imaging enables the automation reliability and time resolution of change detection within and between the image sequences. Production speed and economy of the highly automated system is getting superior as the flood of the redundant raw data is removed that otherwise would drown the image analysts and large-area mapmakers. As a by-product of developing these new DTI technologies, the extended foundations of math and engineering sciences are laid for other applications. They could range from the conceptual loop inverse expansion of the general theory of relativity to highly automated large-scale mapping and remote sensing of the Earth and other members of our solar system and galaxy.

For example, the angular or scale-invariant triangle adjustment of range measurements is allowed to have an unknown scale bias. The constrained loop inverse operators can still provide the unbiased estimators of the angular sub-space of the triangle shape. There is no need to assume that the universe is expanding if this scale bias seemingly increases toward the edges of the universe (like the Sun seemingly rotates the Earth). A defective math model may cause some unknown systematic error near any singular solar or galaxy system when the original observations (time, assumed speed of light, etc) are converted into the local range estimates to define the Gauss-Markov model of a triangle or other 4-D and higher dimensional network adjustment of the solar or galaxy systems. This is analogous to the bias of piecewise geodetic nets before the unified world GPS system could connect the local nets to one master reference datum. We need the “galaxy tie point” observations to notice this bias of Duane Brown’s concept “fool’s paradise” within our solar system for their interconnection to other galaxy datum tie points.

Despite a pile of Duane Brown’s and my papers and proofs of system prototypes, the softcopy WS R&D did not hear/see/grasp the problem of fool’s paradise and was recently joined by another fool, the Lidar technology. Both techniques required costly global edit/merge, feathering and other ad hoc processes to fuse the map on strip/block overlap areas to the ground control or to each other. This can be avoided by self-calibration and densification of the ground control points as a by-product of the proposed DTI system design or using some hyper sensor targeting ideas of urban and city scenes related to the retro and red-dot targeting of (Brown, 1994).

The discussed 4-D DTI concept may also get applicable to the emerging digital stereo TV and movie technology (such as Avatar) by some retro-targeting modifications of Moire technique, including the reflective projector imaging of inverse photogrammetry in industrial metrology of car, airplane and other manufacturing. The integral array algebra has potential new applications in 4-D projector and range sensing systems of stereo GPS, SAR and Lidar. They are dating back to my GLSM related ideas in Meissl’s direct linear (bundle) adjustment theory of GPS range sequences, (Meissl, 1979). It resulted in the direct linear projective normal equations of the bundle method and fast array algebra Replacement Sensor Models as discussed in my 2006 ASPRS convention papers, (Rauhala, 2006a-b). A meeting to discuss these concepts in Florida some 30 years ago was canceled by Meissl’s tragic death in avalanche of Austrian Alps before his planned US trip.

MULTI-LINEAR LOOP INVERSE REPLACEMENT OF TAYLOR EXPANSION

The Lm-inverse of general matrix inverse and estimation theory in array algebra expands the concepts and notations of matrix and tensor calculus. Its full-rank starting idea of 5x5 reseau transform for Hasselblad close-range moon camera of Apollo missions applied a 3x3 grid for separable 1-D interpolation model in the observable space domain, (Rauhala, 1972). This replaced the traditional transform domain parameters of Taylor expansion by inverting their consistent system of equations into the equivalent 2-D interpolation parameters in the space domain. The theory of linear least squares was applied to the observed 5x5 grid as linear predictions from the unknown 3x3 grid. The resulting full-rank least squares filtering solution of the unknown grid is achieved *very fast* by two separable 1-D filtering processes. The adjusted or filtered 3x3 grid can then be transformed back to the adjusted 3x3 transform domain by the consistent inverse Fast Taylor Transform (FTT) if needed. Large grid sizes and uniform weights allow the simulation of separable and shift invariant filter banks for various global vs. local interpolation functions of the multi-linear replacement model of Taylor transform in the observable space domain.

An example of fitting a 2-D FTT polynomial of 3x3 coefficients X to the 5x5 observed values L_{obs} of Hasselblad reseau marks is shown to provide the foundations to LSC and ILSM techniques. The task is to filter the systematic or low pass film deformation from the random (high pass) error of the observed values. The 3x3 sub-grid (containing the center and corner points of the 5x5 observed grid) L_{11} of two times sparser spacing is used as the interpolation model or basis functions in place of the 3x3 Taylor polynomial unknowns in a "fast" least squares adjustment. In terrain modeling, this is analogous to fitting an interpolation grid of four times less number of elevation posts to an overly dense observed grid. This thought evolved into the global finite element replacement technique of the local FTT models in LSM and its automated edit, (Rauhala, 1980b), (Rauhala et. al., 1989).

Let the 1-D interpolation grid L_1 to coincide at a sub set of the variable locations v= -2, 0, +2 of the observed locations v= -2, -1,0, +1,+2 of L_{obs}. The linear observation equations of the 1-D interpolation parameter array L_1 along all $m_2 = 5$ observed columns read in this example of $n_1 = 3$ parameters and $r_1 = 2$ redundant observations

$$\begin{bmatrix} \underset{n_1,n_1}{I} \\ \underset{r_1,n_1}{K_{21}} \end{bmatrix} \underset{n_1,m_2}{L_1} = \begin{bmatrix} \underset{n_1,m_2}{L_{1obs}} \\ \underset{r_1,m_2}{L_{2obs}} \end{bmatrix} + \begin{bmatrix} \underset{n_1,m_2}{V_1} \\ \underset{r_1,m_2}{V_2} \end{bmatrix}, \quad K_{21} = A_{21} A_{01}^{-1}, \tag{1.1}$$

$$\underset{2,3}{K_{21}} = 1/8 \begin{bmatrix} 3 & 6 & -1 \\ -1 & 6 & 3 \end{bmatrix}, \ \underset{2,3}{A_{21}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \underset{3,3}{A_{01}} = \begin{bmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}.$$

These partial 1-D column (vertical) adjustments have 3x5=15 basis functions in L_1. The least squares compression technique LSC in next section will exploit the fact that the partial 1-D column residuals V_2 of (1.1) are typically smaller or more compact than the following, combined column and row residuals, V_{21}, V_{22} of the 2-D polynomial using only 3x3 basis functions (elevation posts) in grid L_{11} of the partitioned array algebra observation equations

$$\begin{bmatrix} I \\ K_{21} \end{bmatrix} \underset{n_1,n_2}{L_{11}} \begin{bmatrix} I & K_{22}{}^T \end{bmatrix} = \begin{bmatrix} \underset{n_1,m_2}{L_{1obs}} \\ \underset{r_1,m_2}{L_{2obs}} \end{bmatrix} + \begin{bmatrix} \underset{n_1,m_2}{V_1} \\ \underset{r_1,m_2}{V_2} \end{bmatrix} = \begin{bmatrix} \underset{n_1,n_2}{L_{11obs}} & \underset{n_1,r_2}{L_{12obs}} \\ \underset{r_1,n_2}{L_{21obs}} & \underset{r_1,r_2}{L_{22obs}} \end{bmatrix} + \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}. \tag{1.2}$$

The least squares solution of the 3x3 2-D grid parameters L_{11} of (1.2) is found in array algebra by applying the shift-variant filter matrices H_1, H_2 as the full-rank L-inverses of its separable design matrices

$$\begin{aligned} \underset{n_1,n_2}{\hat{L}_{11}} &= \begin{bmatrix} I \\ K_{21} \end{bmatrix}^L \underset{m_1,m_2}{L_{obs}} \begin{bmatrix} I \\ K_{22} \end{bmatrix}^{LT} = \underset{n_1,m_2}{\hat{L}_1} \begin{bmatrix} I \\ K_{22} \end{bmatrix}^{LT}, \ m_1 = n_1 + r_1, \ m_2 = n_2 + r_2, \\ &= (I + \underset{n_1,n_1}{K_{21}^T K_{21}})^{-1} [L_{11obs} + K_{21}^T L_{21obs} + (L_{12obs} + K_{21}^T L_{22obs}) K_{22}] (I + \underset{n_2,n_2}{K_{22}^T K_{22}})^{-1} \\ &= \underset{n_1,m_1}{H_1} \underset{m_1,m_2}{L_{obs}} \underset{m_2,n_2}{H_2}{}^T = \underset{n_1,m_2}{\hat{L}_1} \underset{m_2,n_2}{H_2}{}^T, H_1 = (I + \underset{n_1,n_1}{K_{21}^T K_{21}})^{-1} \begin{bmatrix} \underset{n_1,n_1}{I}, \underset{n_1,r_1}{K_{21}^T} \end{bmatrix}. \end{aligned} \tag{1.3}$$

The superscript "L" in (1.3) denotes the full-rank Least squares or L-inverse of the rectangular left and right side design matrices of the 2-D interpolation parameters L_{11}. The resulting space domain solution of the low pass grid can be transformed into the traditional parametric or transform domain FTT solution of least squares by

$$
\begin{aligned}
(1.4)\qquad \underset{n_1,n_2}{\hat{X}} &= \underset{n_1,n_1}{A_{01}^{-1}}\,\underset{n_1,n_2}{\hat{L}_{11}}\,\underset{n_2,n_2}{A_{02}^{-1T}} = \underset{n_1,m_1}{A_1^L}\,\underset{m_1,m_2}{L_{obs}}\,\underset{m_2,n_2}{A_2^{LT}} = \underset{n_1,n_1}{A_{01}^{-1}}(\underset{m_1,n_1}{A_1}\,\underset{n_1,n_1}{A_{01}^{-1}})^L\,\underset{m_1,m_2}{L_{obs}}[\underset{n_2,n_2}{A_{02}^{-1}}(\underset{m_2,n_2}{A_2}\,\underset{n_2,n_2}{A_{02}^{-1}})^L]^T \\
&= (A_1^T A_1)^{-1} A_1^T L_{obs} A_2 (A_2^T A_2)^{-1} = \underset{n_1,m_2}{\hat{Y}}\,\underset{m_2,n_2}{A_2^{LT}},\ \underset{n_1,m_2}{\hat{Y}} = \underset{n_1,m_1}{A_1^L}\,\underset{m_1,m_2}{L_{obs}}, \\
&\underset{3,3}{A_{01}} = A_{02}\ ,\ \underset{5,3}{A_1} = A_2 = \begin{bmatrix} A_{01} \\ A_{21} \end{bmatrix},\ A_1^L = (A_1^T A_1)^{-1} A_1^T, \mathrm{rank}(A_1) = \mathrm{rank}(A_{01}) = n_1 = 3.
\end{aligned}
$$

We have now arrived at the basic 1971 idea of fast array calculus of the multi-linear estimation theory. The least squares inverse of one large 2-D design matrix $A = A_1 \otimes A_2$ of the traditional vector $vec(X)$ of polynomial parameters (where the parameters are stacked into a vector in the way a computer stacks an array) is now replaced by two L-inverses of the small 1-D design matrices A_1, A_2. The tensor product property $\otimes$ of two matrix inverse multiplications is exploited by applying the first multiplication to all observed m_2 columns and then, in corner turning, the second set of 1-D solutions is applied along the row direction of the resulting low pass solutions $\hat{L}_1$ of (1.3) and $\hat{Y}$ of (1.4). The operation count of these two 1-D inverse matrix multiplications is O(N) vs. O(N**3) of the traditional solution of N parameters. The 2-D Fast Fourier Transform is an inflexible special case where the two square design matrices can be split into additional tensor products with 2x2 matrices, (Rauhala, 1976, 1980a).

Before proceeding to the LSC technique of separable condition adjustment, in a more general way than the 2-D wavelets of JPEG2000, the idea of loop inverses and their role in the general theory of estimation is outlined. The basis transform matrices A_{01}, A_{02} between the modeling parameters X and the space domain parameters L_{11} need not be square but have fewer or as many p_1, p_2 independent rows as the ranks of matrices A_1, A_2. Their full-rank minimum norm or m-inverses exist such that (1.4) becomes

$$
\begin{aligned}
(1.5)\qquad \underset{n_1,n_2}{\hat{X}} &= \underset{n_1,p_1}{A_{01}^m}\,\underset{p_1,p_2}{\hat{L}_{11}}\,\underset{p_2,n_2}{A_{02}^{mT}} = \underset{n_1,m_1}{A_1^{Lm}}\,\underset{m_1,m_2}{L_{obs}}\,\underset{m_2,n_2}{A_2^{LmT}} = \underset{n_1,p_1}{A_{01}^m}(\underset{m_1,n_1}{A_1}\,\underset{n_1,p_1}{A_{01}^m})^L\,\underset{m_1,m_2}{L_{obs}}[\underset{n_2,p_2}{A_{02}^m}(\underset{m_2,n_2}{A_2}\,\underset{n_2,p_2}{A_{02}^m})^L]^T \\
&= \underset{n_1,p_1}{A_{01}^m}\,\underset{p_1,m_1}{H_1}\,\underset{m_1,m_2}{L_{obs}}(\underset{n_2,p_2}{A_{02}^m}\,\underset{p_2,m_2}{H_2})^T,\ \ \underset{n_1,p_1}{A_{01}^m} = \underset{n_1,p_1}{A_{01}^T}(\underset{p_1,p_1}{A_{01}A_{01}^T})^{-1},\ \underset{n_2,p_2}{A_{02}^m} = \underset{n_2,p_2}{A_{02}^T}(\underset{p_2,p_2}{A_{02}A_{02}^T})^{-1}.
\end{aligned}
$$

The Lm-inverse produces the Moore-Penrose pseudo-inverse A^+ of matrix A when $p = \mathrm{rank}(A)$ for any full row-rank basis transform matrix A_0 regardless the choice of the location of the space domain parameters L_0. Basis functions $L_0 = A_0 X$ span the observable space always estimable under the Gauss-Markov model $E(V) = 0$. They span some subspaces by choosing $p < \mathrm{rank}(A)$ and expand the basic definitions and concepts of linear and nonlinear estimators and general matrix inverses, (Rauhala, 1974, 1975, 1976, 1979, 1980a, 1981a, 2002a, 2002b).

This general inverse and linear estimation theory caused a controversy as it satisfies only the more general (reflexive) Penrose condition GAG=G vs. the consistency condition AGA=A of the g-inverse. The inverse Taylor derivatives of consistent nonlinear transforms in the Q-surface theory of (Blaha, 1994) show this inherent GAG=G rule prompting the nonlinear Lm-inverse expansion of Taylor transforms of (Rauhala, 2002b) to be introduced in section 3. Section 2 summarizes the idea of LSC from (Rauhala, 2006b) by connecting the linear Lm-inverse (1.5) to the condition adjustment, sequential least squares and wavelets techniques.

LEAST SQUARES COMPRESSION (LSC)

The Lm-inverse predecessor of multi-resolution wavelet theory exploited the least squares condition adjustment known in mathematical surveying for hundreds (perhaps thousands) of years. The connection of the sequential adjustment technique of (Mikhail and Helmering, 1973) in terms of the space domain Lm-inverse parameters provided a common sense solution to the problem of general matrix inverses in (Rauhala, 1974 p.71, 1976, 2002a-b). The 1-D partial adjustment of parameters $L_1 = L_{11}\left[I, K_{22}{}^T\right]$ in (1.1) is now shown using the connection of the sequential adjustment with the Lm-inverse and condition adjustment in the general case $p_1 \le n_1, m_1 = p_1 + r_1$

$$\hat{L}_1 = L_{1obs} - K_{21}{}^T (I + K_{21}K_{21}{}^T)^{-1} w, \text{ where } w = (L_{2obs} - K_{21}L_{1obs}), \qquad \text{seq. adjustment}$$

$$\begin{bmatrix}\hat{V}_1 \\ \hat{V}_2\end{bmatrix} = B^m w = B^T (BB^T)^{-1} w = \begin{bmatrix}-K_{21}{}^T \\ I\end{bmatrix}(I + K_{21}K_{21}{}^T)^{-1} w, \underset{r_1,m_1}{B} = \begin{bmatrix}\underset{r_1,p_1}{-K_{21}} & \underset{r_1,r_1}{I}\end{bmatrix} \text{ cond adjustment.}$$

$\Rightarrow$ Column (vertical) process of separable least squares compression:

(2.1)
$$\underset{r_1,m_2}{\hat{V}_2} = (I + \underset{r_1,r_1}{K_{21}K_{21}{}^T})^{-1} \underset{r_1,m_2}{w},$$
$$\underset{p_1,m_2}{\hat{V}_1} = -\underset{p_1,r_1}{K_{21}{}^T} \underset{r_1,m_2}{\hat{V}_2},$$
$$\underset{p_1,m_2}{\hat{L}_1} = \underset{p_1,m_2}{L_{1obs}} + \underset{p_1,m_2}{\hat{V}_1}.$$

The column process of separable least squares compression in (2.1) reverses the order of the partial least squares solution (1.3) of (1.1). The low-pass solution $\hat{L}_1$ is computed first in (1.3) and the high pass residuals $\hat{V}_2$ are then found by subtracting the observed values from the interpolated values at the redundant locations. The number of 1-D low pass parameters is typically larger than the number of redundant observations so solving the p_1 equations of $\hat{L}_1$ along each column of (1.1) takes more computations than the $\hat{V}_2$ solution of r_1 equations of the condition adjustment in (2.1). Matrix $I + K_{12}K_{12}^T$ of (2.1) has a special structure enabling fast solutions especially when the separable 1-D (linear or cubic) local interpolators use only 2 or 4 neighboring low-pass points.

The partial 1-D compression stage along columns consists of encoding only the high pass residual array $\hat{V}_2$ since array $\hat{V}_1$ can be reconstructed from it as shown in (2.1). All observed values can be reconstructed by encoding and decoding only the high pass least squares residuals of the redundant observations after the reconstruction of the low pass signal $\hat{L}_1$ in the row (horizontal) decoding process. The reversible reconstruction of the observed data is achieved using the decoded low pass and high pass (residual) solutions of the 1-D column processing stage by

(2.2)
$$L_{1obs} = \hat{L}_1 + K_{21}^T \hat{V}_2$$
$$L_{2obs} = K_{21}\hat{L}_1 - \hat{V}_2.$$

There are no restrictions on the size and structure of the interpolator K_{21} to get a reversible reconstruction in (2.2) using the properly encoded/decoded $\hat{L}_1, \hat{V}_2$ of (2.1). For example, linear interpolation or averaging of each redundant observation from two neighboring nodes causes no reconstruction loss compared to using cubic and other more global interpolation models that may produce smaller (more compact) residuals $\hat{V}_2$.

The multi-D compression applies the 1-D technique of (2.1) to each variable in analogy to (1.3) and (1.4). The 2-D low pass solution $\hat{L}_{11}$ of (1.3) is recovered by the 1-D LSC process along the row (horizontal) direction of the partial 1-D solution $\hat{L}_1$ of (2.1). *This 1-D row process need **not** be applied to the column residuals $\hat{V}_2$ of (2.1) to break it into the LH and HH (V_{21}, V_{22}) components of the wavelet theory*. The speed of row encoding and decoding is now doubled from that of the wavelet (n=2m) and 2-D transform coding of n=m. Encoding of residuals $\hat{V}_2$ is already done in the column process. Thus, the amount of encoding is reduced by the factor of three compared to JPEG2000 corner turning as only the high-pass V_{12enc} residuals of the $\hat{L}_1$ row adjustment (HL component) need encoding and residuals V_{11} can be reconstructed from them in analogy to (2.1). Combined with 1-D row interpolations using the decoded $\hat{L}_{11}$, they provide the reversible row reconstruction of $\hat{L}_1$ needed in (2.2) for 1-D column reconstruction of $L_{1obs} = \left[L_{11obs}, L_{12obs} \right]$ and $L_{2obs} = \left[L_{21obs}, L_{22obs} \right]$. Note that V_{11}, V_{12enc} are not equal to the 2-D least squares solution residuals of (1.2), (1.3). Further improved speed and compaction are possible in the industrial (lossy 3-D and 4-D) applications of this theory in the Data-To-Info (DTI) system concept.

NON-LINEAR INVERSE OF FAST TAYLOR TRANSFORM (FTT)

The expansions of differential and integral calculus are needed to provide the Fast Taylor Transform (FTT) inputs of on-board ILSM and LSC. They required the nonlinear array algebra tools of extended matrix and tensor calculus to perform the general parameter transform among the transform domain parameters X and the space domain basis functions L_1 by the inversion of the consistent system of the local differential tensor Taylor expansion

(3.1) $$dL_1 = F_1{}'dX + 1/2F_1{}''dX**2 + 1/6F_1{}'''dX**3 + 1/24F_1^{IV}dX**4....$$

by

(3.2) $$d\hat{X} = K_1{}'dL_1 + 1/2K_1{}''dL_1**2 + 1/6K_1{}'''dL_1**3 + 1/24K_1^{IV}dL_1**4... = dX^0 + ddX + dY.$$

An FdX**n operator denotes repeated vector dX contractions of the n last indices of array F, (Rauhala, 1992, 2002). The linear inverse derivative term G=K' already shows the basic loop inverse property K'F'K'=K' where K' is the m-inverse of F' such that the linear truncation of (3.2) is equivalent to the minimum norm Newton-Gauss (N-G) solution. This reflexive or r-inverse property expands to 3-D in the inverse second derivative expressed by the exponential matrix and array multiplications and multi-linear matrix notations by

(3.3) $$K_1{}'' = k_1{}''(j_1, i_2, i_3) = -F_1{}'^m F_1{}'' F_1{}'^m**2 = -\sum_i g(j_1, i) \sum_{j2} g(j_2, i_2) \sum_{j3} g(j_3, i_3) f_1{}''(i, j_2, j_3).$$

Last two indices of a 3-D array F" in (3.3) are contracted with the first index of matrix G and pre-multiplied by G.

The 4-D 3rd order inverse derivative array K'" has four 4-D components involving the second and third order forward derivatives F", F'" and the first and second order inverse derivatives as shown by (Blaha, 1994) in terms of the mechanical or indical tensor notations. After years of struggle in deriving the 4th and higher order inverse terms I replaced the tensor notations with the more intuitive matrix-like contraction operators of arrays, such as

(3.4)
$$\begin{aligned}
K_1{}''' &= -F_1{}'^m\{F_1{}''' F_1{}'^m**3 + nF_1{}'' F_1{}'^m**1K_1{}''\}, \\
K_1^{IV} &= -F_1{}'^m\{F_1^{IV} F_1{}'^m**4 + n_1F_1{}''' F_1{}'^m**2K_1{}'' + F_1{}''[n_2F_1{}'^m**1K_1{}''' + n_3K_1{}''K_1{}'']\}, \\
K_1^{V} &= -F_1{}'^m\{F_1^{V} F_1{}'^m**5 + n_4F_1^{IV} F_1{}'^m**3K_1{}'' + F_1{}'''[n_5F_1{}'^m**2K_1{}''' + n_6F_1{}'^m**1K_1{}''K_1{}''] \\
&\quad + F_1{}''[n_7K_1{}''K_1{}''' + n_8F_1{}'^m**1K_1^{IV}]\}.
\end{aligned}$$

The order of "full contractions" (where all free indices are contracted) with the exponential powers of vector dL in (3.2) turn the arrays into matrices until the end result becomes a vector. The integer multipliers n depend on the number of permutations of the indices to be contracted. For example, n=3 in the second term of K'" in (3.4) for triple contractions of (3.2) where the last index of F" is contracted by the N-G solution vector dX=GdL. The result F"dX**1 is a matrix for multiplying a vector after the two last indices of K" are contracted with vector dL.

Insertion of (3.3) and (3.4) to (3.2) provides the "direct" or closed-form solution of inverse Taylor Transform. The reader should attempt to develop at least the few low order terms to appreciate the more intuitive array notations of (3.4) over the mechanical tensor notations. Combining the linear N-G term with the second derivative term of (3.3) and with the leading elements of all terms in (3.4) we get the corrected N-G iteration rule of (Rauhala, 2002b)

$$\begin{aligned} dX_1 &= F_1'^m\, dL_1 - F_1'^m\{1/2F_1''\,dX^0**2 + 1/6F_1'''\,dX^0**3 + 1/24F_1^{IV}\,dX^0**4...\} \\ &= dX^0 + ddX,\ dX^0 = F_1'^m\, dL_1 = F_1'^m[L_{1obs} - F_1(X^0)], \\ ddX &= -F_1'^m[F_1(X^0 + dX^0) - F_1(X^0) - F_1'\,dX^0] = F_1'^m[L_{1obs} - F_1(X^0 + dX^0)]. \end{aligned} \tag{3.5}$$

The correction ddX includes the leading effect of all N-G neglected second and higher order inverse derivatives of Taylor expansion with the benefit that its computation is negligible as the N-G derivative of the first iteration at the point of expansion is reused. The most cumbersome part of deriving the direct solution consists of finding the effect of the remaining terms of the high order inverse derivatives (3.4) upon (3.2). This task requires tedious accounting of terms and contraction rules of Taylor array polynomials with powers of the sum of vectors, (Rauhala, 2002), to identify the following sub Taylor expansions of ddX + dY

(3.6)

$$\begin{aligned} &dY = -F_1'^m\{[F_1'(X^0 + dX^0) - F'][ddX + dY] \\ &+1/2F_1''(X^0 + dX^0)[ddX + dY]**2 + 1/6F_1'''(X^0 + dX^0)[ddX + dY]**3 + ...\} \\ &= ddX + dY - F_1'^m\{[F_1'(X^0 + dX^0)ddX + 1/2F_1''(X^0 + dX^0)ddX**2 + 1/6F_1'''(X^0 + dX^0)ddX**3 + ... \\ &+F_1'(X_2)dY + 1/2F_1''(X_2)dY**2 + 1/6F_1'''(X_2)dY**3 + ...\},\ \ X_2 = X^0 + dX^0 + ddX, \\ &\Rightarrow 0 = ddX - F_1'^m\{F_1(X_2) - F_1(X^0 + dX^0) + F_1'(X_2)dY + 1/2F_1''(X_2)dY**2 + 1/6..\}. \end{aligned}$$

The last line of (3.6) is multiplied with F' and the unknown terms moved to the left-hand side resulting in

$$\begin{aligned} &F_1'(X_2)dY + 1/2F_1''(X_2)dY**2 + 1/6F_1'''(X_2)dY**3 + ... = L_{1obs} - F_1(X_2) \\ &dY^0 = F_1'(X_2)^m[L_{1obs} - F_1(X_2)], \\ &ddY = F_1'(X_2)^m[L_{1obs} - F_1(X_2 + dY^0)], \\ &\hat{X} = X_2 + dY^0 + ddY \Leftrightarrow F_1(\hat{X}) = L_{1obs}. \end{aligned} \tag{3.7}$$

In other words, (3.7) repeats another corrected N-G iteration of (3.5) at the updated point of expansion.

The theory of nonlinear matrix inverses is started by a Taylor expansion of the reflexive m-inverse of F' by

(3.8)

$$
\begin{aligned}
F_1{}'^{mr} &= K_1' + K_1'' dL_1 **1 + 1/2 K_1''' dL_1 **2 + 1/6 K_1^{IV} dL_1 **3 + 1/24 K_1^{V} dL_1 **4 + \ldots \\
&= K_1' - F_1{}'^{m} \{ F_1'' dX^0 **1 + 1/2 F_1''' dX^0 **2 + 1/6 F_1^{IV} dX^0 **3 + 1/24 F_1^{V} dX^0 **4 + \ldots \\
&-1/2[2F_1'' dX^0 **1 F_1{}'^{m} F_1'' dX^0 **1 + F_1''(F_1{}'^{m} F_1'' dX^0 **2)**1] + 1/6[n_1 F_1''' dX^0 **2 K_1'' dL_1 **1 + \ldots\} F_1{}'^{m} \\
&= F_1{}'^{m} - F_1{}'^{m} \{ \Delta' - \Delta' F_1{}'^{m} \Delta' + \Delta' F_1{}'^{m} \Delta' F_1{}'^{m} \Delta' - \ldots \} F_1{}'^{m} \\
&= F_1{}'^{m} - F_1{}'^{m} (I + \Delta' F_1{}'^{m})^{m} \Delta' F_1{}'^{m} = F_1{}'^{m} - F'(\hat{X})^{m} \Delta' F_1{}'^{m} = F_1'(\hat{X})^{m}, \quad \Delta' = F'(\hat{X}) - F_1', \\
\hat{X} &= X^0 + d\hat{X}, \; d\hat{X} = dX^0 + ddX + dY^0 + ddY, \; dX^0 = F_1{}'^{m} dL_1.
\end{aligned}
$$

The inverse derivative (3.8) at the solution (3.7) of (3.2) expands the theory of a single scalar quadratic or cubic equation, a starting point of the LSM theory in its 2-D least squares expansion. It resolves the classical problem of finding all or closest real roots of 4th and higher order scalar polynomials. The paper and pencil derivations of these FTT inverses in the LSM application prompted the early invention of 1-D (line match) ILSM where the scalar normal equation of x-shift is found by integrating the product of Taylor expansions of the nonlinear derivative and residual functions over the interval of a symmetric window around the point of expansion. This allows the window size to be any fraction of a pixel and yet the number of discrete observables is infinity. Elimination of the linear bias term (with no a priori weights) makes the reduced x-shift normals into a cubic or 4th degree polynomial with a direct solution of the scalar quadratic and cubic technique of the corrected N-G in (3.7-8).

The flaw of missing terms in the N-G technique gets obvious in (3.2) - (3.8). It is corrected in a "super-iteration" of solving a consistent nonlinear system – without any Taylor truncation or rank restriction of (Schnabel and Franks, 1984). This is achieved with the correction term ddX of (3.5) by evaluating the constant Taylor term at N-G solution and repeating the N-G step using the same derivative matrix as in the first step, and then repeating these two steps with the updated derivative for corrections dY and ddY. This super iteration is further expanded in a "hyper iteration" of nonlinear Lm-inverse by inserting the inverse solution (3.2) to the forward Taylor expansion of actual observations. The solution of the least squares normal equations of the space domain Taylor parameters dL is combined with its inverse mapping into the adjusted transform domain parameters X, (Rauhala, 2002b). An equivalent (or better) solution is found by expressing the nonlinear condition adjustment in terms of the residual parameters V to provide the adjusted values L_{1obs} of (3.7) for the direct consistent nonlinear solution of adjusted X.

Recent work (started over 20 years ago) has focused on the elusive "fast" solution of the nonlinear inverse problem of FTT where the Taylor terms X of (1.4) form 2-D or 3-D arrays of LSC in the ILSM and RSM applications. A nonlinear tensor decomposing by array Cholesky of (Rauhala, 1975, 1976) combines the LSC technique with FTT of the reference image and EBA results of the ILSM output as discussed under the DTI system concept. The 2-D FTT wavelet expansion serves the input of ILSM pipeline processes of the mentioned closed-form paper and pencil derivation of its output that then feeds the input of EBA fusion and automated validation of the ILSM registrations. A recent review of multi-linear tensor decomposing can be found in (Kolda and Bader, 2009). The integral expansion of nonlinear inverse FTT solution of ILSM cannot be detailed in this introductory paper.

LEAST SQUARES MATCHING AND CORRELATION

The direct solution of FTT closes a circle in the theory of LSM inventions since its early work, (Rauhala, 1977). The theory started from the 1-D quadratic equation (1.1) for 1-D line matching to find such a shift dx of reference line gray value function f(dx)=1/2f" dx**2 + f' dx + c that it equals the average gray value g of the slave line. No adjustment is needed assuming that the first and second order derivatives of the slave line near the selected point of Taylor expansion equal those of the reference line and that there is no systematic radiometric error in c and g of the average gray values. The quadratic solution of possible two real roots has the obvious advantage that the first order derivative f' can be close to zero – and the closed form square root solution of both roots is found without any iteration! The N-G technique would get lost even with a slight error in the observed averages c and g.

The estimation theory or adjustment calculus used in photogrammetry and geodesy has no or little use for solving consistent systems of linear or nonlinear equations without redundant observations of proper a priori weights and error propagation to estimate the variance-covariance matrix of adjusted parameters and their functions. Thus, the line LSM model has to include a linear bias correction db to utilize the N-G technique of error propagation. To get one redundant observation with the quadratic Taylor terms we have to measure all three gray values in the slave line. The normal equation of the shift parameter dx becomes linear after elimination of the additive bias term db, reading f" dx= g'-f' when no a priori magnitude weights are applied and when no Taylor truncation of the nonlinear observation equations are made, (Rauhala, 1992).

The match points at "adverse areas" of small first derivatives or large db values (indicating changed or dissimilar scene contents) require an automated on-line edit by proper weights based on the error propagation of past iterations and a new local variance estimate of the least squares residuals over an entire window. The known estimates of slopes in both directions of the window for both dx and dy shift parameters can be eliminated by reshaping the slave window values such that the constant bias and shift correction parameters are valid at the central pixel locations of a relatively small (3x3–3x5 pixel) window and dense spacing (2x2-3x3 pixels). Good initial values from the previous iterations of the LSC image pyramid feed the automated Global LSM (GLSM) edit and validation of the local x-shift normals in fast finite element network adjustment, (Rauhala, 1980b, 1986), (Rauhala et.al., 1989).

GLSM treats the weighted local normal equations of LSM x-shifts as the observed values L_{obs} of the elevation grid in (1.3). The redundant observations consist of properly weighted continuity equations of finite elements, first along one direction and then after separable or non-separable "corner turning" along the other direction. The continuity constraints automatically fill or predict some optimal local pull-in values at the points of poor weights using the neighboring observed values of optimal least squares weights. This network solution is fast (million pts/sec in PC). Large continuity weights can be applied at the highly minified levels of image pyramid to pull-in the large global shifts of the entire slave image. This cures the pull-in problem of LSM such that its high resolution and typical precision of 0.05-0.2 pixels can be exploited. The resulting terrain models and contour lines are optimally smoothed requiring no or less manual edit than the non-global techniques, (Hermanson et. al., 1993)

Prototype testing with the global least squares correlation of an additive bias term provided a comparable quality and speed to GLSM in 1989. Independent work in the Pressure Sensitive Paint (PSP) technique of wind tunnel testing by Dr. Ruyten confirmed the superiority of least squares correlation over the traditional normalized cross correlation, still a standard in many industries, (Ruyten, 2002). Our 2006 joint proposal proved the equality of LSM with the N-G solution of maximized cross correlation of a multiplicative bias correction, (Rauhala, 2006a), (Rauhala and Ruyten, 2008). Similar rejected proposals since mid 1970's have produced the new math and DTI system foundations of this paper for starting the industrial work of advanced 4-D imaging and range sensing systems.

SUMMARY

A review of the early multi-linear expansion of matrix and tensor calculus was made with a tale of the struggles during the past 20-30 years in its nonlinear expansion of differential and integral calculus. A story of past proposals for its Data-To-Info system application of data fusion and advanced automation of 4-D imaging and range sensing provided some idea about the potential applications of the new math. The basic idea of array algebra Taylor expansion and Least Squares Compression provided an introduction to the latest derivation of inverse Taylor expansion and its application in Least Squares Matching and correlation of 4-D imaging and range sensing.

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