COVARIANCE PROPAGATION FROM SPECIFIC TO GENERIC MODEL

Henry J. Theiss, Senior Scientist
Sensor Geopositioning Center
National Geospatial-Intelligence Agency (IAI, contractor)
12300 Sunrise Valley Dr
Reston, VA 20191
Henry.j.theiss.ctr@nga.mil

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ABSTRACT

Geopositioning from optical sensors onboard airborne platforms, e.g. UAVs, requires a sensor model and subsequently a means to perform rigorous covariance propagation of position and attitude parameters to the ground space. Builders of such imaging systems often do not know precisely what elements to report, and how to report them, in the data stream along with the image pixel data. As such, it is mutually beneficial – to both builders of an imaging system and to exploiters of the imagery – to define a minimum set of metadata elements, including associated covariance information, which shall be populated with values. This paper presents an example specific physical sensor model containing a position offset vector between GPS antenna and perspective center, and a series of two attitude measuring devices, i.e. inertial navigation system measurements of platform attitude and gimbal angle measurements with respect to the platform. The paper presents a generic frame sensor model in which the stochastic information is defined by the standard set of six exterior orientation parameters, i.e. three elements of absolute sensor position and three elements of sensor attitude with respect to an image coordinate system. The equations are provided to map full covariance information associated with each parameter of the specific physical sensor model to the condensed generic physical parameter set. A Monte Carlo Analysis is performed to verify the derived equations. Finally, an analysis is performed to quantify the amount of correlation that exists between position and attitude elements of the generic physical parameter set as a function of the specific sensor geometry. By providing an example specific physical sensor model, and associated equations to transform covariance information to a generic model, this paper provides the methodology from which one can derive the analogous covariance propagation equations for essentially any specific sensor model.

INTRODUCTION

The objective of this paper is to provide the equations required to map covariance matrices of the individual error components of a frame imaging system to a full 6 by 6 covariance matrix associated with a standard frame camera sensor model. The top left 3 by 3 is the covariance matrix associated with position and lower right 3 by 3 is associated with the standard three angles, $\delta\omega$, $\delta\varphi$, $\delta\kappa$, representing attitude errors about the x, y, z axes, respectively, of a frame coordinate system. These three angles represent the collective effects of several other orientation angles on which will be elaborated in the following sections.

COORDINATE SYSTEMS

Figures 1 and 2 illustrate the coordinate systems involved in a typical airborne optical frame imaging system, namely Geocentric (g), North-East-Down, or NED (n), Platform (p), Sensor (s), and Record (r). Such a configuration is consistent with the SENSRB and EG0801 documents. The NED and platform systems have the same origin at the center of navigation. When all platform angles (heading, pitch, and roll) are zeros, these two systems are coincident. Occasionally, a local object coordinate system, East-North-Up, or ENU, is used in place of the Geocentric system. However, for this development the Geocentric is selected because of being the desired standard.

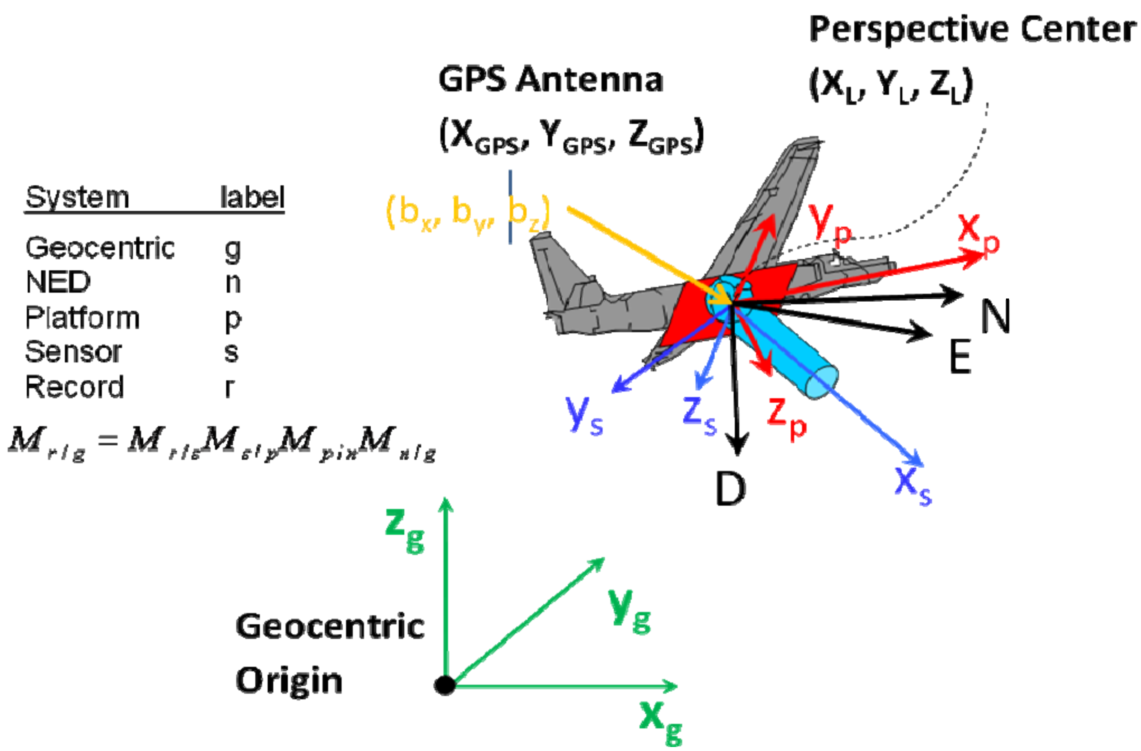

Figure 1. Coordinate Systems overlaid on aircraft.

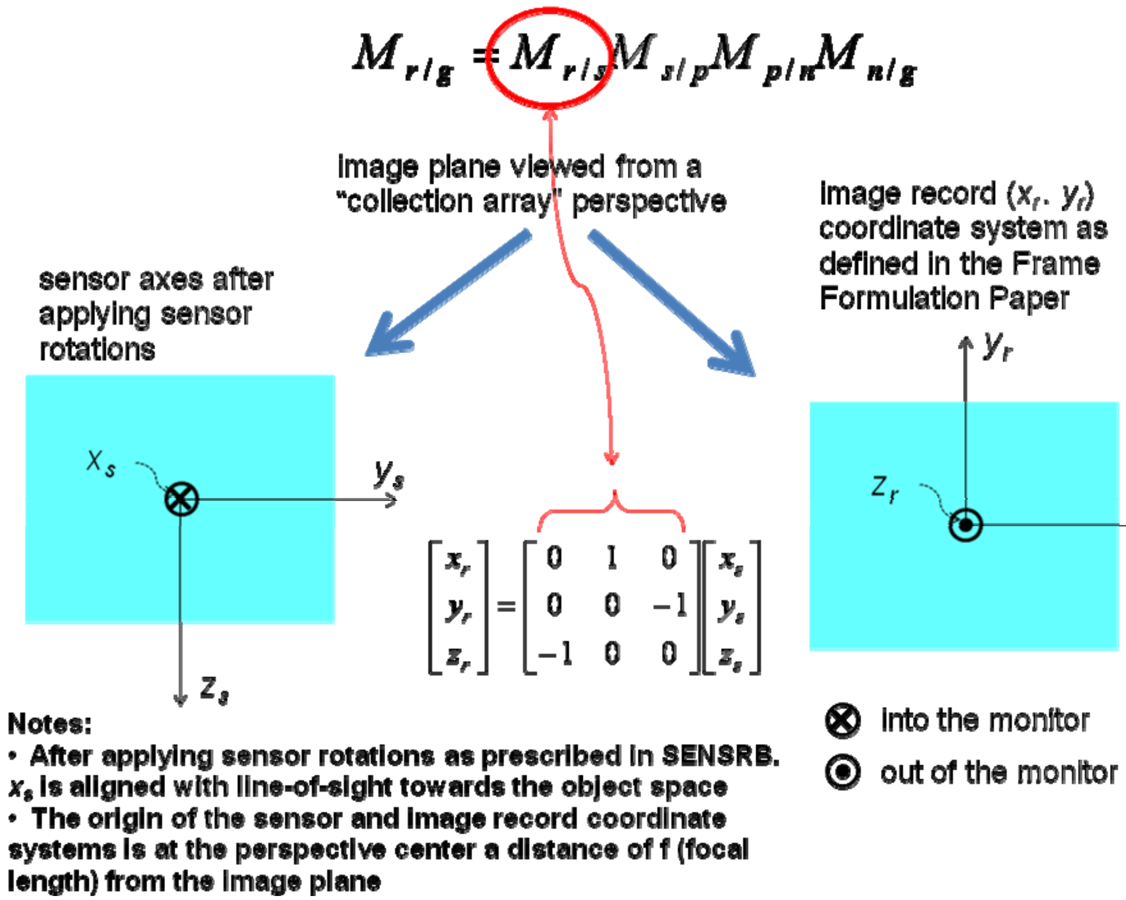

Figure 2. Sensor and Image Record Coordinate Systems as viewed on a monitor.

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PROJECTION MODEL

The following derivation will, in general, use the matrix notation $M_{b/a}$ to designate an orthogonal matrix that rotates coordinate system "a" until it is parallel with coordinate system "b". The collinearity equation can be written as follows:

$$\begin{bmatrix} x \\ y \\ -f \end{bmatrix} = kM \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix} = kM_{r/s} M_{s/p} M_{p/n} M_{n/g} \begin{bmatrix} X - X_L \\ Y - Y_L \\ Z - Z_L \end{bmatrix} \quad (1)$$

where x,y are image coordinates (shifted to the principal point and corrected for all systematic errors) in a Record Coordinate System (RCS); f is the focal length; k is a unique scale factor per ground point; X,Y,Z are ground coordinates in a Geocentric, or Earth-Centered-Earth-Fixed (ECEF), Coordinate System (GCS); and X_L, Y_L, Z_L are the coordinates of the camera perspective center in the GCS; $M_{r/s}$ is the orthogonal rotation matrix that aligns the sensor coordinate system to the record coordinate system; $M_{s/p}$ aligns the platform to the sensor coordinate system; $M_{p/n}$ aligns the NED to the platform coordinate system; and $M_{n/g}$ aligns the Geocentric to the NED coordinate system.

As shown in Figure 2, the camera perspective center location is a function of the GPS antenna location, the base (aka lever arm) vector from the origin of the GPS antenna to the perspective center, and the platform attitude with respect to the GCS. Note that we selected, as an example, the case where the perspective center (L) is also at the origin of the NED (and the platform) coordinate system or center of navigation. In Figure 1 of Section 2.1 of the Frame Formulation Paper, the general case is shown where there is another offset vector from the platform origin to the perspective center. The coordinates of the perspective center in the GCS are given by:

$$\begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} = \begin{bmatrix} X_{GPS} \\ Y_{GPS} \\ Z_{GPS} \end{bmatrix} + M_{n/g}{}^T M_{p/n}{}^T \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix} \tag{2}$$

where bx, by, bz are the components of the base vector measured in the Platform Coordinate System. By substituting Equation 2 into Equation 1, we obtain:

$$\begin{bmatrix} x \\ y \\ -f \end{bmatrix} = kM_{r/s}M_{s/p}M_{p/n}M_{n/g}\left(\begin{bmatrix} X - X_{GPS} \\ Y - Y_{GPS} \\ Z - Z_{GPS} \end{bmatrix} - M_{n/g}{}^T M_{p/n}{}^T \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix}\right) \tag{3}$$

STOCHASTIC MODEL

In order to establish the covariance mapping equations, we must define the stochastic models for the standard frame camera sensor model and then for the example frame imaging system.

The stochastic model for the standard frame involves re-formulating Equation (1) such that it isolates the random variation to six adjustable parameters with zero expected values (these six parameters correspond to the standard six Exterior Orientation elements of a frame image); the middle part of Equation (1) is re-written as follows:

$\underset{3\times1}{a} = k\underset{3\times3}{(\delta M)}\underset{3\times3}{M}\underset{3\times1}{A}$, which in expanded form becomes:

$$\begin{bmatrix} x \\ y \\ -f \end{bmatrix} = k\begin{bmatrix} 1 & \delta\kappa & -\delta\varphi \\ -\delta\kappa & 1 & \delta\omega \\ \delta\varphi & -\delta\omega & 1 \end{bmatrix} M \begin{bmatrix} X - (X_L + \delta X_L) \\ Y - (Y_L + \delta Y_L) \\ Z - (Z_L + \delta Z_L) \end{bmatrix} \tag{4}$$

in which $M = M_{r/s}M_{s/p}M_{p/n}M_{n/g}$. The combined effect of all the component matrices, as represented by M, is the three "generic" sequential angles, ω, φ, κ, which can be extracted from the elements of M, if needed. The attitude errors, all of which have zero expected value, are then manifested by the three terms, δω, δφ, δκ.

The stochastic model for the example frame imaging system involves re-formulating Equation (3) such that it isolates each error component as follows:

$$\begin{bmatrix} x \\ y \\ -f \end{bmatrix} = k \underset{3\times3}{T} \underset{3\times1}{u} \tag{5a}$$

where T and u are a temporary matrix and vector, respectively, used to break the long equation into two separate pieces as follows:

$$T = M_{r/s} \begin{bmatrix} 1 & 0 & -\delta R_p \\ 0 & 1 & 0 \\ \delta R_p & 0 & 1 \end{bmatrix} M_{2R} \begin{bmatrix} 1 & \delta R_h & 0 \\ -\delta R_h & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_{3R} \begin{bmatrix} 1 & \delta I_h & -\delta I_p \\ -\delta I_h & 1 & \delta I_r \\ \delta I_p & -\delta I_r & 1 \end{bmatrix} M_{p/n} M_{n/g} \tag{5b}$$

where M_{2R} and M_{3R} are the rotation matrices that are a function of gimbal resolver measurements in sensor pitch and heading, respectively; i.e., $M_{s/p} = M_{2R}M_{3R}$. The R stands for resolver and the 2 and 3 correspond to the axis of rotation, i.e. about the current Y and Z axes, respectively.

$$u = \left(\begin{bmatrix} X - \left(X_{GPS} + \delta X_G\right) \\ Y - \left(Y_{GPS} + \delta Y_G\right) \\ Z - \left(Z_{GPS} + \delta Z_G\right) \end{bmatrix} - M_{n/g}{}^T M_{p/n}{}^T \begin{bmatrix} 1 & \delta I_h & -\delta I_p \\ -\delta I_h & 1 & \delta I_r \\ \delta I_p & -\delta I_r & 1 \end{bmatrix}^T \begin{bmatrix} b_X + \delta b_X \\ b_Y + \delta b_Y \\ b_Z + \delta b_Z \end{bmatrix} \right) \tag{5c}$$

The terms $\delta X_G, \delta Y_G, \delta Z_G$ are error terms in the GPS platform position, and δb_X, δb_Y, δb_Z are error terms associated with the offset base; both sets are additive. (All six error terms have zero expectations, and two finite 3 by 3 error covariance matrices, Σ_{GG} and Σ_{BB}, respectively). Errors in the platform orientation are matrix multiplicative and are given by the symbols $\delta I_h, \delta I_p, \delta I_r$, in Equation (5c). (The three error terms have zero expectations and a 3 by 3 covariance matrix, Σ_{II}). The error contributors can be grouped into vectors of random variables, GPS (G), INS (I), base (B), and gimbal resolver (R), as follows:

$$\underset{3\times1}{l_G} = \begin{bmatrix} \delta X_G & \delta Y_G & \delta Y_G \end{bmatrix}^T$$

$$\underset{3\times1}{l_B} = \begin{bmatrix} \delta b_X & \delta b_Y & \delta b_Z \end{bmatrix}^T$$

$$\underset{3\times1}{l_I} = \begin{bmatrix} \delta I_r & \delta I_p & \delta I_h \end{bmatrix}^T$$

$$\underset{2\times1}{l_R} = \begin{bmatrix} \delta R_p & \delta R_h \end{bmatrix}^T$$

$$\underset{11\times1}{l} = \begin{bmatrix} \underset{1\times3}{l_G{}^T} & \underset{1\times3}{l_B{}^T} & \underset{1\times3}{l_I{}^T} & \underset{1\times2}{l_R{}^T} \end{bmatrix}^T$$

Note in Equation 5b that the INS angle errors can be modeled in a combined matrix since the navigator performs calculations in an inertial system based on IMU measurements and Kalman Filtering; hence the output of such calculations is an attitude error covariance referenced to the current platform coordinate system. However, the resolver angle errors need to be modeled in separate error covariance matrices since each angle measurement is made sequentially.

At a high level, we can summarize the covariance propagation required to map from example imaging system to standard frame system as follows:

$$\begin{aligned} E &= f_1\left(P\,,\,A\right) = f_1\left(l_G\,,\,l_B\,,\,l_I\,,\,l_R\right) = f_1\left(l\right) \\ P &= f_2\left(l_G\,,\,l_I\,,\,l_B\right) \\ A &= f_3\left(l_I\,,\,l_R\right) \end{aligned} \tag{6}$$

where E, P, A represent exterior orientation, position, and attitude, respectively, of the standard record system; and f1, f2, and f3 symbolically represent functions. Since the INS components appear in both P and A, clearly the covariance propagation will result in a full 6 by 6 covariance matrix, i.e. representing correlation between position and attitude.

We can now apply the general error propagation equation to the first line of Equation (6) as follows:

$$\underset{6\times6}{\Sigma_{EE}} = \underset{6\times15}{J_{El'}}\,\underset{15\times15}{\Sigma_{l'l'}}\,\underset{15\times6}{J_{El'}}^{T} = \begin{bmatrix} \underset{3\times3}{\Sigma_{PP}} & \underset{3\times3}{\Sigma_{PA}} \\ \underset{3\times3}{\Sigma_{PA}}^{T} & \underset{3\times3}{\Sigma_{AA}} \end{bmatrix} \tag{7}$$

where Σ_{PP} is the position covariance matrix, Σ_{AA} is the attitude covariance matrix, and Σ_{PA} is the cross-covariance matrix between position and attitude.

Note that the 15 by 1 vector *l'* is referenced in this equation instead of the 11 by 1 vector *l*. We need to introduce fictitious observations with zero values and zero errors in order to facilitate the covariance propagation. When a gimbal resolver measures an angle, e.g. in heading, it is known that the rotation and associated precision of the angles in pitch and roll will be zeros; hence the placeholders associated with Rp and Rr were zeroed out in the second expanded matrix of Equation (5b). Similarly when the gimbal resolver measures the pitch, it is known that the rotation and associated precision of the angles in heading and roll will be zeros; hence the placeholders associated with Rh and Rr are zeroed out in the first expanded matrix of Equation (5b).

We can expand the Jacobian matrix in Equation (7) as follows:

$$\underset{6\times15}{J_{El'}} = \begin{bmatrix} \underset{3\times3}{J_{PG}} & \underset{3\times3}{J_{PB}} & \underset{3\times3}{J_{PI}} & \underset{3\times6}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{J_{AI}} & \underset{3\times6}{J_{AR}} \end{bmatrix}$$

where the Jacobian sub-components corresponding to position can be obtained by referencing Equations (2) and (5c) as follows:

$$J_{PG} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J_{PB} = M_{n/g}{}^{T} M_{p/n}{}^{T}$$

$$J_{PI} = \begin{bmatrix} J_{PIr} & J_{PIp} & J_{PIh} \end{bmatrix}$$

$$J_{PIr} = M_{n/g}{}^T M_{p/n}{}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix}$$

$$J_{PIp} = M_{n/g}{}^T M_{p/n}{}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix}$$

$$J_{PIh} = M_{n/g}{}^T M_{p/n}{}^T \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix}$$

and the Jacobian sub-components corresponding to attitude can be obtained by referencing Equations (4) and (5b) as follows:

$$J_{AI} = M_{r/s} M_{s/p}$$

$$J_{AR} = \begin{bmatrix} \underset{3\times3}{M_{r/s}} & \underset{3\times3}{M_{r/s} M_{2R}} \end{bmatrix}$$

The covariance matrix for the 15 by 1 vector *l'* can be constructed as follows:

$$\underset{15\times15}{\Sigma_{l'l'}} = \begin{bmatrix} \underset{3\times3}{\Sigma_{GG}} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times6}{0} \\ & \underset{3\times3}{\Sigma_{BB}} & \underset{3\times3}{0} & \underset{3\times6}{0} \\ & & \underset{3\times3}{\Sigma_{II}} & \underset{3\times6}{0} \\ sym & & & \underset{6\times6}{\Sigma_{RR}} \end{bmatrix}$$

The $\sum_{GG}$, $\sum_{BB}$, and $\sum_{II}$ matrices are in general full 3 by 3 covariance matrices provided in the image metadata. The 6 by 6 covariance matrix $\sum_{RR}$ would be constructed as a function of the elements of a full 2 by 2 covariance matrix of resolver angles as follows:

$$\Sigma_{RR} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & \sigma_{Rp}^{2} & 0 & 0 & 0 & \sigma_{RpRh} \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ sym & & & & & \sigma_{Rh}^{2} \end{bmatrix}$$

where σ_{Rp}^{2}, σ_{Rh}^{2}, σ_{RpRh} are the variance of pitch resolver measurement, variance of heading resolver measurement, and covariance between pitch and heading resolver measurements, respectively.

MATLAB EXAMPLE

Synthetic Frame Image:
Focal length = 152mm
Flying height = 1000 m AGL = 1000 m HAE
4 check points, one at each corner of a 100mm by 100mm frame
Base (GPS to perspective center lever arm components, meters) = 15, 11, -12
Platform heading, pitch, roll (deg) = 40, -15, 13
Sensor heading, pitch (deg) = 45, -50 (note: -90 deg is nadir when platform is level)
Input Precisions:
Image coordinate sigmas = 0.015mm
Check point height sigmas = 1 m

GPS covariance (meters squared): $\Sigma_{GG} = \begin{bmatrix} 4 & 1 & 1 \\ & 4 & 1 \\ sym & & 9 \end{bmatrix}$

Base covariance (meters squared): $\Sigma_{BB} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ & 1 & 0.5 \\ sym & & 1 \end{bmatrix}$

INS covariance (radians squared): $\Sigma_{II} = \begin{bmatrix} 0.0002 & 0.00008 & 0.00005 \\ & 0.0001 & 0.00006 \\ sym & & 0.0001 \end{bmatrix}$

Gimbal resolver covariance (radians squared): $\Sigma_{resolver} = \begin{bmatrix} 0.00005 & 0.00002 \\ sym & 0.00006 \end{bmatrix}$

Note that the magnitudes of some of the numbers are unrealistic, e.g. the base vector and the existence of correlation between resolver angles, but did not want to assume diagonal matrices in order to fully test the theory.

Note that the magnitudes of the elevation angles (90 degrees minus the off-nadir angle) for check points 1 through 4 were 60, 32, 57, and 30 degrees, respectively.

For these four check points, the following output provides a comparison of the 3 by 3 ground coordinate covariance matrix derived using the two different techniques: 1) via the two-step process of mapping the sensor-specific 11 by 11 covariance matrix to the generic 6 by 6 covariance matrix (Equation 7) and then performing standard covariance propagation through a generic frame sensor model's image-to-ground function; and 2) via the process of performing standard covariance propagation through the specific sensor model's image-to-ground function directly. Then, the results are shown for the case assuming a block diagonal of 3 by 3 sub-matrices, i.e. ignoring correlation between position and attitude.

Results using full 6 by 6 versus direct error prop.:

1)	220.618170037311	-40.9940694361992	0.271504504153547
	-40.9940694361992	352.766249870072	-0.540769287713719
	0.271504504153547	-0.540769287713719	1.00010607734182
	220.618170037241	-40.9940694361544	0.271504504153541
	-40.9940694361544	352.766249869869	-0.540769287713701
	0.271504504153541	-0.540769287713701	1.00010607734182
2)	1208.92820880467	184.733458783294	-1.53820779310716
	184.733458783294	469.045473082101	-1.06294413660742
	-1.53820779310723	-1.06294413660742	1.0008688509752
	1208.92820880457	184.733458783221	-1.53820779310712
	184.733458783221	469.045473082005	-1.06294413660738
	-1.5382077931072	-1.06294413660739	1.0008688509752
3)	190.677807214114	-128.486508018024	0.190176456243646
	-128.486508018024	352.427637189316	0.647247263698036
	0.190176456243646	0.647247263698035	1.00013330292768
	190.677807214043	-128.486508017952	0.190176456243651
	-128.486508017952	352.427637189112	0.647247263698018
	0.190176456243651	0.647247263698018	1.00013330292768
4)	2359.69170296035	-1048.46783817954	-2.14793032601751
	-1048.46783817954	1076.8600142002	1.28693774835313
	-2.14793032601759	1.28693774835317	1.00113771021422
	2359.69170296007	-1048.46783817926	-2.14793032601753
	-1048.46783817926	1076.86001420002	1.28693774835314
	-2.14793032601748	1.28693774835308	1.00113771021422

Results assuming a block diagonal of 3 by 3 sub-matrices:

1)	221.778172948054	-42.0725039341913	0.271634601261149
	-42.0725039341913	363.131015549378	-0.541643820765941
	0.271634601261149	-0.541643820765941	1.00010615259039
	220.618170037241	-40.9940694361544	0.271504504153541
	-40.9940694361544	352.766249869869	-0.540769287713701
	0.271504504153541	-0.540769287713701	1.00010607734182
2)	1199.87542860969	186.91711583948	-1.53671337553236
	186.917115839481	467.609993108896	-1.06316837872444
	-1.53671337553241	-1.06316837872445	1.0008685838565
	1208.92820880457	184.733458783221	-1.53820779310712
	184.733458783221	469.045473082005	-1.06294413660738
	-1.5382077931072	-1.06294413660739	1.0008688509752
3)	192.886019869503	-131.510885686274	0.189947663645153
	-131.510885686274	361.577818631608	0.648040083681862
	0.189947663645153	0.648040083681862	1.00013337253685
	190.677807214043	-128.486508017952	0.190176456243651
	-128.486508017952	352.427637189112	0.647247263698018
	0.190176456243651	0.647247263698018	1.00013330292768

4)	2351.97152640524	-1050.11540157101	-2.14637694948784
	-1050.11540157101	1072.41412797991	1.28669965521726
	-2.146376949488	1.28669965521725	1.00113731841628
	2359.69170296007	-1048.46783817926	-2.14793032601753
	-1048.46783817926	1076.86001420002	1.28693774835314
	-2.14793032601748	1.28693774835308	1.00113771021422

COMMENTS

1. As shown in the overall and the components results, the covariance mapping technique provides essentially equivalent results to direct error propagation of the individual components.
2. The differences appear to be due to only round-off errors.
3. The differences between the two methods are significant (beyond round-off errors) in the case where the 6 by 6 exterior orientation covariance matrix is treated as a block diagonal (two 3 by 3 blocks, i.e. ignoring the cross covariance matrix Σ_{PA}, in Equation 7).
4. While the approach in this appendix addressed the case where the GPS antenna is offset from the camera perspective center, it can be extended to handle other cases, e.g. where the origins of the platform and gimbal systems do not coincide with the perspective center.
5. The covariance mapping technique presented in this appendix provides the additional benefit of, as a by-product, defining a reduced set of adjustable parameters for a sensor model in a standard reference frame.

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