OBLIQUE PHOTOGRAPHS FOR THE SURVEYOR

R. M. Wilson

EDITOR'S NOTE: This is another chapter for the Manual of Photogrammetry which the Society is having prepared by outstanding leaders in this specialized field of engineering.

THE principles described in this paper will interest the surveyor or engineer who wishes to make practical use of the wealth of information displayed in oblique photographs. Whether the camera used be one of the precision instruments of formal photogrammetry, or an ordinary inexpensive hand camera, the principles are the same in application. The purpose may be to extend reconnaissance mapping, to develop auxiliary control for detailed topographic mapping, or to make a close-up dimensional study of details in a construction project from "photographic memoranda." In practice the graphical method requires only the ordinary drafting equipment already available to any engineer.

Oblique photographs, as referred to here, are those taken from either air or ground stations with the camera pointed up or down at any angle of elevation or depression preferably not greater than about 45° from the horizon. Such photographs are too often regarded merely as pretty pictures of the landscape, interesting only because they present a general view in the kind of perspective most familiar to the eye. But the information they contain may be interpreted accurately and with but little effort in terms of true horizontal and vertical angles, useful in determining distances and differences of elevation.

In previous articles[1,2] the writer has described the photoalidade, which provides an adjustable holder for the photograph and a telescopic sighting device similar to that of a transit. When the photograph has been properly placed with respect to the axes of this instrument, sights may be taken readily to details shown upon it just as they would be taken to actual features in the real landscape. But how many surveyors can have one of these instruments conveniently at hand? The need for a fully graphical method was pointed out by Col. Thomas North, and he has published[3] a description of a simple procedure that should find wide application. His method uses angles transferred from the plane of the photograph through the isocenter to the map plane. These angles are useful for rapid radial-line compilations, although they are measured at a point that is not near the nadir. The writer has been tempted to use the first part of the method described by Colonel North, but to develop it according to the somewhat different principles involved in operating the photoalidade, employing true horizontal angles measured at the nadir. The result contains ideas that are closely related to those already developed[4] by Capt. D. R. Crone, of the Survey of India, in a solution of the problem which has been referred to as the "Indian Method." However, the following procedure seems so simple and direct as to deserve presentation in its own right, and it is believed that an entirely new technique is used in adjusting the observations to obtain the most probable results.

[1] Wilson, R. M., A new Photoalidade: *The Military Engineer*, November–December 1937.

[2] Wilson, R. M., Oblique Photographs and the Photoalidade: PHOTOGRAMMETRIC ENGINEERING, issue of April–May–June, 1938.

[3] North, Col. Thomas, Locating Points from Oblique Airphotos—: *Field Artillery Journal*, September–October 1940.

[4] Jenney, Capt. R.C.N., R. E., The Indian Method of Compiling High Oblique Air Photographs and its Routine Use in Determining Spot Heights for Contouring Vertical Air Photographs: *Photogrammetria*, 1939, Vol. II No. 3.

THE PROBLEM

The object is to determine the true horizontal angles at the camera station as subtended between points or features on the ground that are shown on the photographs and, also, to determine the true vertical angles of depression or elevation to these points from the camera station. These are the angles that might be measured with a transit—on the limb and on the vertical arc—if the station could be occupied with a transit instead of a camera.

Just as in using a transit, it is necessary to know or to find the position of the station from which the observations are taken, and to orient the observations with respect to known directions before they can be used to determine new directions, positions, or elevations. The simple special case, when the camera has been leveled and its axis pointed horizontally in a known direction from a known position, is described in many text books on surveying. The representation of the true horizon line then passes through the center of the photograph and through the fiducial marks at the middle of the right and left edges of the photograph if the camera is equipped and adjusted to show these correctly.

Generally, however, oblique photographs are not taken with a leveled camera. Therefore the true horizon line may be neither through the center of the photograph, nor parallel to its edges or its geometric axes. At a ground station, sights might be taken with a leveling instrument to find points in the field of view which lie in the plane of the true horizon; these points, identified on the photograph, would provide the means to draw the horizon line there. But auxiliary observations of this kind cannot be made from camera stations in the air.

The general problem, therefore, involves finding the horizon line on the photograph, and then determining the position, elevation and orientation of the camera, entirely through information that appears in the photograph itself. Usually the solution is possible if at least three ground points of known position and elevation can be identified in the photograph. These control points on the ground compose the base upon which the solution rests, just as a surveyor's tripod rests firmly upon the sharp points of its three legs. Obviously they should form a triangle as large and wide as is feasible within the field of view, in order to attain maximum stability in the resulting solution. If the three points are in line the solution will fail, just as the tripod would not stand alone with the points of its three legs set all in line.

FOCAL LENGTH

When taking the photograph the rear nodal point of the camera lens is the station point of perspective for the view as projected upon the sensitized film or plate held in the focal plane. If the camera is focussed at infinity, the length of the perpendicular dropped from the nodal point to the focal plane is the focal length of the lens. It is assumed that this perpendicular coincides with the axis of the lens. The foot of the perpendicular determines the principal point of the photograph. But in the present problem, when using a paper print, it is necessary to know the effective distance, f', from the plane of that print to its own perspective station point. This is not likely to be exactly the same as the focal length of the lens, unless the photograph is a contact print, made without expansion or contraction in the paper or the negative, and from an exposure made with the camera focussed at infinity. A finished print may be compared with its negative and, if it has changed in size or scale, a proportionate correction may be applied to the known focal length of the lens to obtain the distance required. Prints made glossy by "squeegee" drying or by "tinning" are often so badly distorted as to be useless for the purposes described here. If the lens was drawn out a little to focus on near objects, the amount of that movement should be added as a further correction. Although such corrections suffice for ordinary purposes, a more formal means of calibrating the camera, prints, or diapositives should be employed if the extreme precision of this method is to be attained.

General Relationships

Referring to the perspective diagram, Figure 1, suppose that the photograph is held with its perspective station coinciding with the camera station at C and

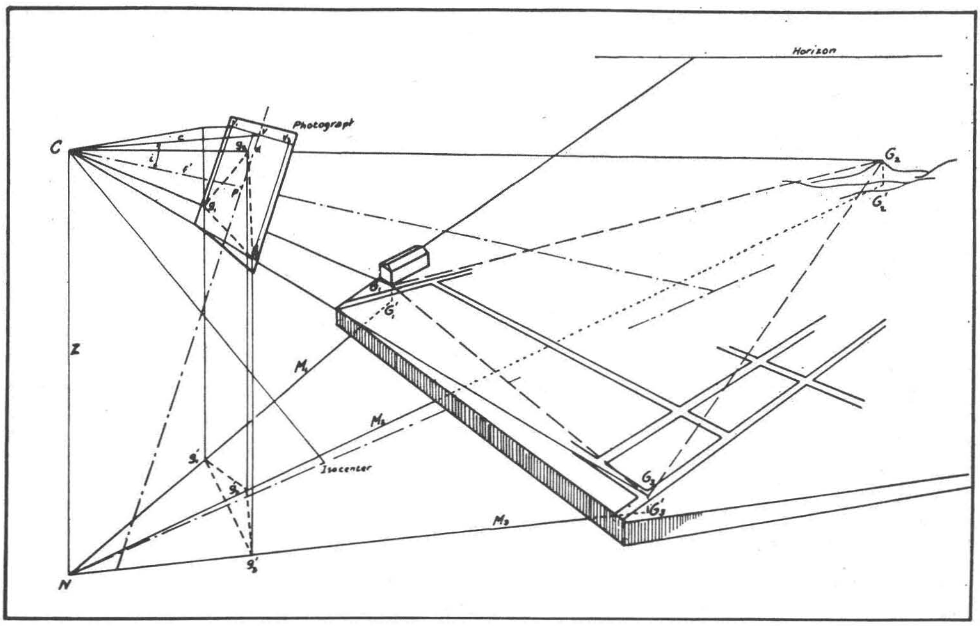


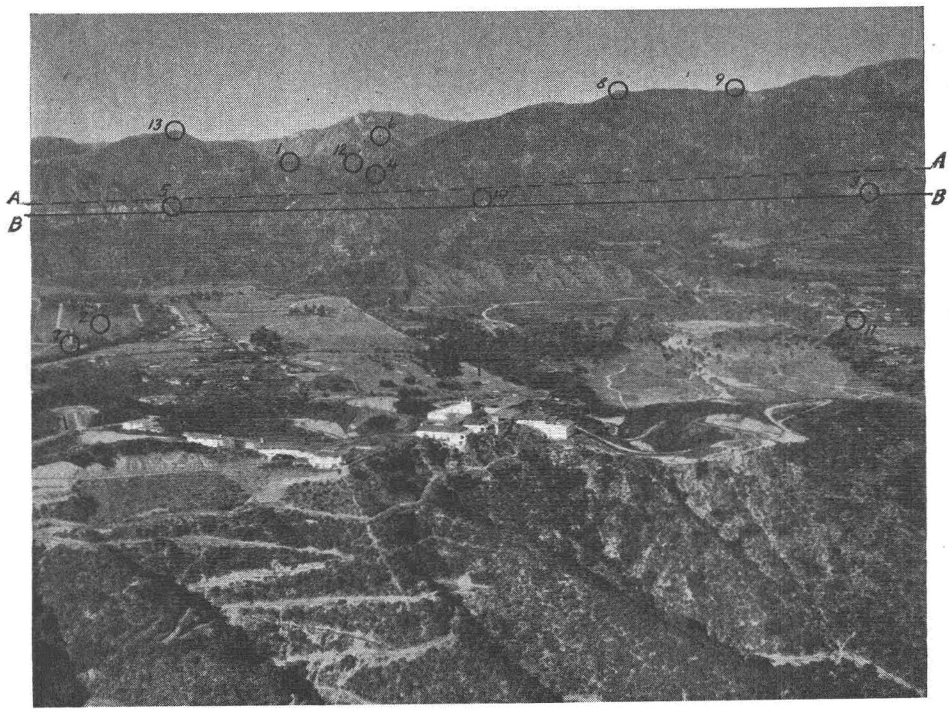
Fig. 1

is also adjusted to the particular angles of tilt, swing and azimuth that represent the circumstances at the instant of exposure. The objects G_1, G_2, G_3, shown as the corner of a building, a hilltop triangulation station, and a street intersection, respectively, are the three ground control points required for the solution; it is assumed that accurate position and elevation are known for each of these points. From the corners near G_1 and G_3 the photographed area spreads wider and wider with distance, and in this example its length stretches out to the horizon. The plumb line dropped down from the camera station reaches the horizontal datum plane at the nadir, N, and the distance, Z, is the altitude of the camera.

The rays from C to G_1, G_2, G_3 will pass through g_1, g_2, g_3 exactly where the images of those objects appear on the photograph. The line v_1v_3 is the trace of the horizontal plane through C upon the plane of the photograph; it is the line to be determined on the photograph that will represent the true horizon of the camera station.

Part of the problem is to measure the horizontal angles between directions observed from C to features in the field of view. Plumb lines dropped to the datum plane from the ground features G_1, G_2, G_3, and others, would determine the projections of those points at G_1', G_2', G_3', etc. Then the lines from N through the projected points indicate the true directions in the datum plane, and angles between them are equal to the corresponding horizontal angles that are to be measured, by means of the photograph, at C.

Similarly, the points g_1, g_2, g_3 and others projected down from the sloping photograph to the datum plane at g_1', g_2', g_3' fall, respectively, upon the horizontal lines of direction already referred to that radiate from N to G_1', G_2', G_3', etc., and which are the projections of the sight rays from C. These same points might be considered also as projected up to the horizontal plane containing C. It is apparent, therefore, that the horizontal angles sought can be found by projecting upon a horizontal plane the information contained in the photograph while the photograph is held in its natural sloping position. The horizon line v_1v_3 is the record on the photograph defining the swing and tilt of that position.



Fairchild Aerial Surveys, Inc., L. A.

FIG. 2

PREPARATION FOR GRAPHICAL CONSTRUCTION

The first step in the procedure of interpretation is to find the horizon line on the photograph. This may be done correctly and at once in the special cases already mentioned, but in general it can be drawn only tentatively at first, by estimation. The apparent horizon often appears in the photograph and is an excellent guide to the estimation. The true horizon is approximately $58.82''\sqrt{Z}$ above the sea horizon in seconds of vertical angle;[5,6] or, in inches on the photograph, $(f'\sqrt{Z} \div 3507 \cos^2 i)$ in which Z is elevation of camera in feet, f' is in inches,

[5] Canada Topographical Survey, A Graphical Method of Plotting Oblique Aerial Photographs: *Bulletin of the Topographical Survey* 1928, Department of the Interior, Dominion of Canada.

[6] Canada Topographical Survey. The use of Aerial Photographs for Mapping: *Topographical Survey Bulletin* No. 62, 1932, Department of the Interior, Dominion of Canada.

and i is the angle of inclination of the camera axis as determined below. Subsequent calculations will indicate the amount of error in the estimation, and provide the corrections to determine the true horizon line. But in order to demonstrate the geometry of the problem it will be assumed that the true horizon line is already drawn correctly instead of only tentatively as shown by A–A, Figure

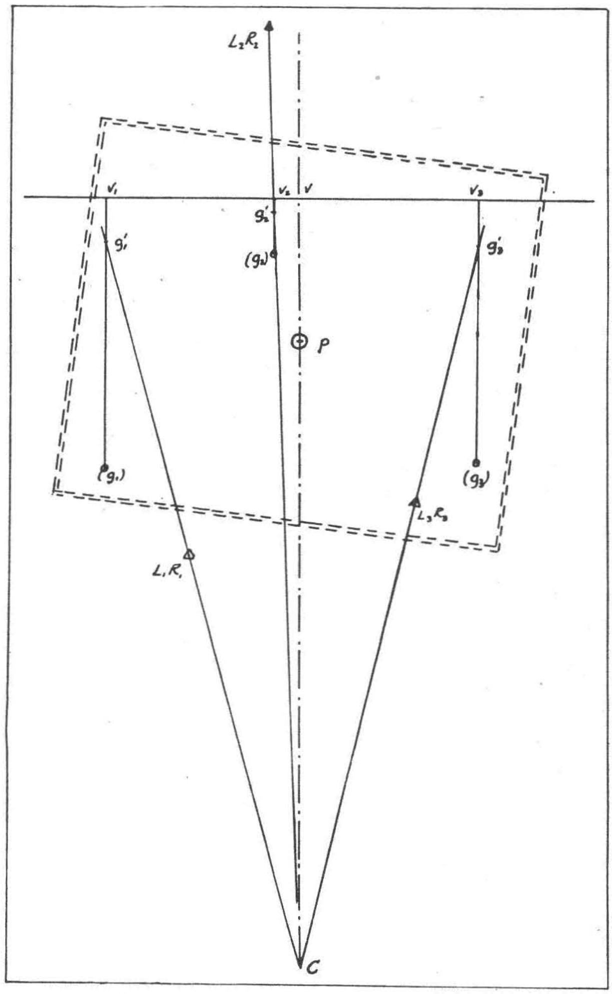


FIG. 3

2. Carefully measure the perpendicular distance from p to v, which will be called d, and divide it by f'; the result is tan i. If d is down from the horizon it is negative and then tan i, and i, are also negative. Find i and cos i, to determine the hypothenuse, from C to v, which will be called c, in the triangle Cvp, Figure 1. The same units, generally either inches or centimeters, should be used for all these lengths.

Now lay a piece of tracing paper over the photograph, as shown in Figure 3, making it large enough to provide, later, for plotting the distance c below the horizon line. Also it may be found desirable for later operations to have the tracing paper extend far enough above the horizon line to accommodate distances from C to points of interest in the field of view as reduced to the scale of the map or work sheet upon which the compiling of information is to be done. The tracing paper should be fastened over the photograph in a way to prevent relative movement, although provision may be made for lifting it at one side for clear inspection of the photograph.

Horizontal Angles

With these arrangements made, trace the line v_1v_3 and construct the perpendicular to it that passes through the traced center of the photograph, (p). Extend this perpendicular, as indicated, to C at the computed distance c from the horizon line (Figure 3). The points on the photograph, representing features

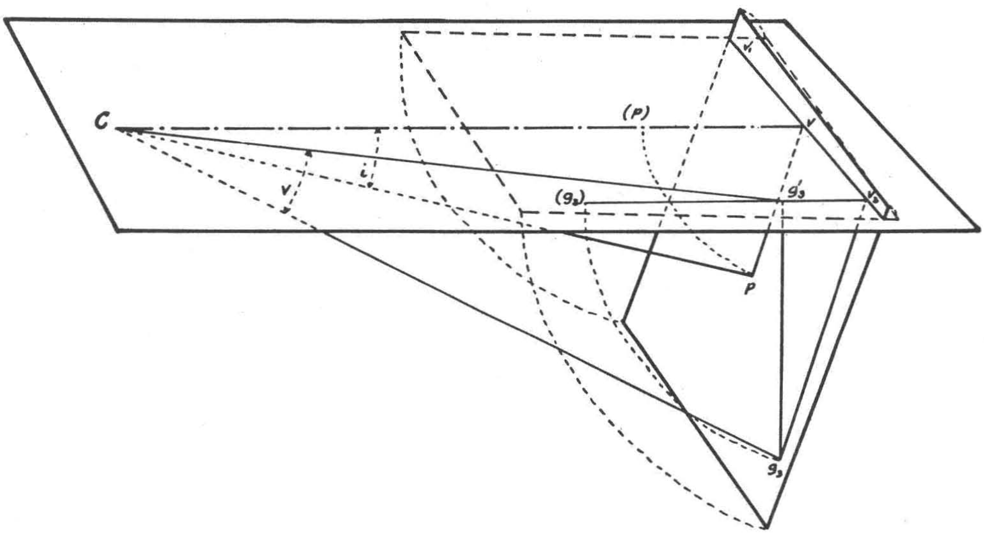


Fig. 4

upon which observations are desired, are traced through on the paper as (g_1), (g_2), (g_3), etc. From each of these points draw a perpendicular to the horizon line. Set a pair of proportional dividers (or use a scale and a slide rule) to the ratio $d:c$ and plot g_1', g_2', g_3', etc., so that each of the perpendiculars will be divided into parts in the same ratio as c is divided by point (p). Now draw the lines of direction from C through g_1', g_2', g_3', etc., which define the required horizontal angles. The tracing paper may be used to transfer these angles to the map or compilation sheet.

The geometric proof of this procedure may be seen in Figure 1, or perhaps more clearly in the locally enlarged view, Figure 4. The horizontal plane through C, and the natural plane of the photograph, are shown intersecting in the horizon line v_1v_3. The photograph as actually used, however, lies on the drafting table with a sheet of tracing paper fastened over it, which is to be considered as a horizontal plane. Its center at (p), and points upon it such as (g_1), (g_2), (g_3) are marked through upon the tracing paper. Now it may be imagined that the photograph is rotated downward about the horizon line v_1v_3 as axis, to its natural position, swinging its center and other points on it down (or up) along the indicated arcs. Then point g_3' is the point on the tracing paper directly above the natural position of g_3. Now the triangle $g_3'v_3g_3$ lies in a plane that is parallel to the principal vertical plane which contains the triangle Cvp. Both are right triangles, and the angle at v in one is equal to the angle at v_3 in the other; therefore the triangles are similar. The hypothenuse of each of these triangles is represented individually on the tracing paper by a line perpendicular to the line v_1v_3. The dividers were set to the ratio $d:c$, because the distance from v to $(p)=d$. Therefore:

$$\begin{aligned} d:c &= \text{short leg} : \text{hypothenuse} && \text{(in triangle } Cvp) \\ &= v_3g_3' : v_3g_3 && \text{(in similar triangle } g_3'v_3g_3) \\ &= v_3g_3' : v_3(g_3) \end{aligned}$$

Thus the distance v_3g_3' can readily be plotted to determine g_3' which is the projection of g_3 upon the horizontal plane. Any points so determined upon the tracing paper are in fact parts of the orthometric projection of the slanting photograph upon a horizontal plane. Therefore the angles between them, about C as a center, are the true horizontal angles sought. It will be convenient to use the letter y for the distance v_3g_3 measured on the photograph, making its value negative when measured down from the horizon line.

Vertical Angles

To measure the vertical angle at C, a somewhat similar proportional method might be employed to determine graphically the projection of v_3g_3, or y, into a vertical plane. It is simpler, and more precise, however, to scale y directly from the photograph and multiply it by cos i to obtain the vertical distance g_3g_3'. Then scale the horizontal distance Cg_3' from the tracing paper, and designate it by the letter m, to use with g_3g_3' in finding the tangent of the vertical angle, V, thus:

$$\tan V = \frac{y \cos i}{m}.$$

The use of this formula and others to follow is illustrated later in a numerical example. Both y and m must be measured in the same units, and both might be measured from the tracing paper. But to avoid the inaccuracies of tracing, it is best to measure y directly from the photograph where the image can be seen clearly; accuracy is required in this measurement because it is the critical one in determining the difference of elevation. The angle V, from C to g_3, is the vertical angle to the ground point G_3 because the sight ray through g_3 continues through to G_3. Generally it will not be necessary to look up the angle itself, because only tan V, already available, is needed in solving for the difference of elevation. Tan V (and V) should be negative in sign if V is down from the horizon.

Resection for Camera Position

Having established the method of measuring horizontal and vertical angles from the photograph, the next step in a practical problem is to find the position, elevation, and orientation of the camera with respect to the ground control.

The positions of the control points being known, they may be plotted accurately on the map or compilation sheet. The horizontal angles between them, as viewed from the camera, are available on the tracing paper referred to in the preceding paragraphs. Therefore, the tracing paper may be laid over the compilation sheet and shifted about until each of the lines of direction radiating from C passes through the plotted position of its own control point. This is the usual and well known tracing paper resection.[7] With the tracing paper so adjusted, the position of C is pricked through with a needle to the compilation sheet to mark the nadir of the camera station; also, the positions of the control points are marked on the tracing paper, to show them there at proper distances from C, on the scale of the compilation sheet. Now the horizontal distances, M, from the camera station to the control points can be scaled either from the compilation sheet or from the tracing paper. These distances, with the tangents of the corresponding vertical angles already available, are used in solving right triangles to find differences of elevation.

Datum for Elevations

The true datum for elevations is the curved sea-level surface of the earth. In this problem, however, it will be less confusing to refer elevations to a datum plane which is exactly horizontal at the nadir of the camera and tangent to the curved sea-level surface there. At all other places this datum plane will be above the sea level surface. Refraction, which is the curving of lines of sight through the atmosphere, really is not concerned in this relationship; it is customary to assume sight lines to be straight, and to compensate for that assumption by using a smaller curvature correction. Accordingly, the combined effect of curvature and refraction should be applied at once to the elevations, E, of all ground points to refer them to the simple datum plane. Such translated elevations, E', will always be less than the actual elevations above sea level; occasionally they may be so much less as to be negative in sign. The relation is expressed by the formula: $E' = E - (\text{curv. \& refr.})$.

The usual expression for curvature and refraction, $.574\ (\text{miles})^2$ may be transformed to $.00002059\ M^2$, using thousand-foot units for M and for the resulting correction also. The curvature and refraction table given here has been computed accordingly.

Certain approximations have been admitted into this step of the procedure. There may be a figure slightly different from the constant .574 that would represent more closely the curvature and refraction for high and long observation lines from an airplane. The direction of a computed difference in elevation is perpendicular to the reference plane; this is not exactly the same as the true vertical direction through the control point along which the difference of elevation should properly be measured. These approximations do not introduce appreciable errors in the results obtained.

Differences of Elevation

The formula for the elevation, Z', above the datum plane, for a point on the reference plane just over an observed ground point is:

$$Z' = E' - M \tan V$$

$$= E' - \frac{My \cos i}{m} \, .$$

The distance, y, and the vertical angle, V, usually will be negative, or down from the horizon. The horizontal distance to the control point, M, is always positive. It will be convenient to express Z', E', and M all in thousand-foot units; y and m in inches or centimeters.

[7] Wilson, R. M. Strength of Three-Point Locations: *Field Engineers Bulletin* No. 11, U. S. Coast and Geodetic Survey, December 1937.

CURVATURE AND REFRACTION

Distance and Correction both in thousand-foot units
$C \& R = .00002059\ M^2$

M	$C \& R$	M	$C \& R$	M	$C \& R$	M	$C \& R$
0	.0000	30	.0185	60	.0741	90	.1668
1	.0000	31	.0198	61	.0766	95	.1858
2	.0001	32	.0211	62	.0791	100	.2059
3	.0002	33	.0224	63	.0817	105	.2270
4	.0003	34	.0238	64	.0843	110	.2491
5	.0005	35	.0252	65	.0870	115	.2723
6	.0007	36	.0267	66	.0897	120	.2965
7	.0010	37	.0282	67	.0924	125	.3217
8	.0013	38	.0297	68	.0952	130	.3480
9	.0017	39	.0313	69	.0980	135	.3753
10	.0021	40	.0329	70	.1009	140	.4036
11	.0025	41	.0346	71	.1038	145	.4329
12	.0030	42	.0363	72	.1067	150	.4633
13	.0035	43	.0381	73	.1097	155	.4947
14	.0040	44	.0399	74	.1128	160	.5271
15	.0046	45	.0417	75	.1158	165	.5606
16	.0053	46	.0436	76	.1189	170	.5951
17	.0060	47	.0455	77	.1221	175	.6306
18	.0067	48	.0474	78	.1253	180	.6671
19	.0074	49	.0494	79	.1285	185	.7047
20	.0082	50	.0515	80	.1318	190	.7433
21	.0091	51	.0536	81	.1351	195	.7829
22	.0100	52	.0557	82	.1384	200	.8236
23	.0109	53	.0578	83	.1418	205	.8653
24	.0119	54	.0600	84	.1453	210	.9080
25	.0129	55	.0623	85	.1488	215	.9517
26	.0139	56	.0646	86	.1523	220	.9966
27	.0150	57	.0669	87	.1558	225	1.0424
28	.0161	58	.0693	88	.1594	230	1.0892
29	.0173	59	.0717	89	.1631	235	1.1371

THE SLOPING REFERENCE PLANE

The difference of elevation ($M \tan V$) computed from each control point may be regarded as the length of the line extending up or down from that point to the reference plane used in measuring the vertical angle. It must be remembered that this reference plane is approximately horizontal and passes through the camera station; it should not be confused with the datum plane that is exactly horizontal at N. If the reference plane were exactly horizontal the ends of such vertical lines from the several control points would all be just at the same elevation—the elevation of the plane and of the camera station. Then all values of Z' would be equal to Z.

But unfortunately the plane probably is not horizontal. It will be recalled that, at the beginning of a practical problem, only an estimated or tentative horizon line can be drawn on the photograph to represent the reference plane. Observations on the photograph referred to the tentative horizon line result in angles measured in, and downward (or possibly upward) from, the corresponding approximately horizontal plane. Such angles miss being true horizontal and vertical angles, respectively, because of the tilted reference system. The effect of this tilt upon horizontal angles is relatively small, and the position determined for the camera station, using the tentative reference plane, is not likely to be greatly in error. But vertical angles may contain the full amount of that tilt and be so seriously affected as to yield widely different figures of elevation for points on the reference plane.

Care has been taken to explain that the elevation, Z', determined from a control point through the scaled distance and preliminary vertical angle is not the elevation at the camera station; it is the elevation of a point on the tentative reference plane just over the control point. Therefore, widely different figures of elevation for different points on the tentative plane are to be expected, because it is known to be only approximately horizontal.

However, after the elevations and positions have been found for three points in the plane, if they are not in line, the slope of the plane can be determined and the error existing in the initial estimation of the horizon line will be disclosed. This information can be used to level up the reference system, and to draw the true horizon line correctly on the photograph.

Correcting the Reference Plane

While the tracing paper was still adjusted over the compilation sheet, the positions of the plotted control points were traced. Now, on the tracing paper at each of these points, write the elevation of the tentative plane as determined there. The slope of the tentative plane may be represented graphically by drawing its contours of elevation on the tracing paper, just as is done on a contour map to represent a sloping ground surface. The contours for the plane must be parallel, equally-spaced straight lines, fitted to the three spotted elevations. Then it is easy to scale off the components of slope in the direction of the principal vertical plane in which the camera was pointed and in the (transverse) direction at right angles thereto. This method has been fully described already,[2] and it will not be expanded again here. Instead, a more recently devised analytical method will now be described.

It is convenient to use a system of rectangular coordinates with its origin at N, measuring Z upward, L horizontally forward and R to the right with respect to the pointing of the camera. The equation for a plane in this system may be written thus:

$$Z' = Z + aL + bR$$

in which a is the component of slope of the plane in the direction forward from the camera, b, the component toward the right, and Z is the particular value of Z' where the vertical axis of the system intersects the plane, that is, the elevation at the camera station in the present problem.

An observation upon a control point from the camera station provides the value of Z' for the spot on the plane which has the same horizontal coordinates as the control point, L and R, which may be scaled from the tracing paper. These three known coordinates are then substituted into the general equation of the plane. The same thing is done for the other control points, so that three observation equations are available to solve simultaneously for the three unknown constants Z, a and b. Using values thus found for a and b, which define the slope of the tentative plane, small distances above or below the ends of the tentative horizon line can be calculated to find points through which the true horizon line should be drawn. These distances are found by the formula:

$$e = \frac{ac + bw}{\cos i}.$$

The distance along the horizon line from v to the place where e is to be plotted—in the side margin of the photograph for example—is represented by w, which should be in the same units used for other measurements on the photograph, and should be positive to the right of v and negative to the left. Where the sign of e is positive, the tentative horizon is above the true horizon line.

All of this procedure, which may seem complicated in the description, may be clarified by a numerical example. The photograph (Figure 2) has been selected for this purpose because some of the identified points on it are above the horizon, and the use of the algebraic signs in the vertical measurements can so be illustrated. The horizon line A–A was chosen simply by estimation. The small distances $+.122$ inches and $+.268$ inches can be plotted in the margins, -4.5 inches to the left and $+4.5$ inches to the right of v, respectively. In this example they are both positive, therefore the tentative horizon line is above the true horizon at both ends. The new horizon line B–B is then drawn below the tentative line, as indicated.

Referred to Estimated Horizon Line $A-A$

$f'=11.583$ $\quad d=1.705$ $\quad c=11.708$

$\tan i=-.14720$ $\quad i=-8°22'25''$ $\quad \cos i=.98934$

Point	Feet ÷ 1,000 Scaled from Tracing			Inches Scaled from		Feet ÷ 1,000			Ft. r
	L	R	M	Photo y	Tracing m	$\frac{My \cos i}{m}$	$E'=$ $E-$Curv. & Ref.	Z'	
1	25.21	−3.79	25.49	+0.325	11.89	+0.689	3.114−.013	2.412	
2	9.51	−3.08	10.00	−1.192	12.13	−0.972	1.194−.002	2.164	
3	17.55	+5.86	18.50	−0.215	12.32	−0.319	2.128−.007	2.440	

Simultaneous observation equations:

$$
\begin{array}{lllll}
Z & +25.21a & -\ 3.79b & -2.412 & =0\\
Z & +\ 9.51a & -\ 3.08b & -2.164 & =0\\
Z & +17.55a & +\ 5.86b & -2.440 & =0\\
\hline
 & +15.70a & -\ \ .71b & -\ .248 & =0\\
 & -\ 8.04a & -\ 8.94b & +\ .276 & =0\\
 & -15.70a & -17.46b & +\ .539 & =0\\
\hline
 & & -18.17b & +\ .291 & =0
\end{array}
$$

$$Z=\ 2.056;\ a=+\ .0165;\ b=+\ .0160$$

Computation of e; in left margin $W=-4.5$, right, $W=+4.5$

$$e=\frac{+.0165(11.708)+.0160(\mp 4.5)}{.98934}=.195\mp .073$$

$$=+.122 \text{ in left margin} \qquad +.268 \text{ in right margin}$$

The position and pointing of the camera represented by the position of C, the rays from that point through the image points on the photograph, and the ground points to which the rays go, compose a rigidly fixed system. A tentative reference plane was passed through that fixed system; this should be replaced now by the new plane that is more nearly horizontal. This leveled-up reference plane does not make any change in the actual directions and slopes of the rays observed from C, it is simply in a better position than the old one to measure those directions and slopes as true horizontal azimuths or bearings and as vertical angles from the true horizon, respectively.

When the new horizon line has been drawn, the old tracing paper should be discarded. Improved determination of horizontal and vertical angles should be made on a new tracing paper, the resection for the position of the nadir should be repeated, and new differences of elevation calculated from the control points to check the elevation of the camera station.

Theoretically, the new measurements to the three control points should result in three exactly equal figures of elevation for the reference plane above the datum plane. But, practically, due to the unavoidable inaccuracies in scaling and drafting, perfect agreement will seldom be achieved. Usually it is fruitless to draw a third horizon line hoping to eliminate any very small residual slope that may be found in the adjusted reference plane. There is a method, to be described later, whereby the effect of such a residual slope can easily be eliminated from the results of observations made upon the photograph.

Application of results

New points, required for the objective purposes of the survey, may now be selected and observed upon. Lines of direction to them, obtained by means of the tracing paper, may be plotted on the compilation sheet. Similar operations with a photograph taken from another camera station will provide intersecting lines to locate the new points in position. The camera stations should be situated so as to avoid sharp intersections. Vertical angles from one or both of the camera stations with scaled horizontal distances, provide the means to determine the elevations of the new intersected points. This is the same method that is employed with a plane table.

Related Instrumental Methods

The graphical method for extracting information from oblique photographs requires only the equipment that every engineer has already on his drafting table.

With a photoalidade, which uses the same principles mechanically, the horizontal and vertical angles can be obtained directly, just as they are found with a plane table and telescopic alidade. But the horizon line must still be determined by trial and correction, the same as in the graphical method. Horizontal angles obtained with the photoalidade have a higher degree of precision than those obtained by graphical construction on the tracing paper; on the compilation sheet, however, they must be used graphically. The great advantage of the photoalidade is its speed and convenience, once the true horizon line has been drawn and the photograph properly adjusted in the instrument.

A still more precise method is available if the coordinates of images on the photograph can be measured closely with a comparator. From these coordinates the horizontal and vertical angles can be computed accurately, without any graphical operation to weaken the result. The horizontal angles obtained may still be laid off on tracing paper and used graphically if so desired, but better advantage may be taken of their inherent precision by using them to compute triangulation instead. Customary triangulation methods should be used for computing "three-point fixes" and for the positions intersected from camera stations so established.[8]

If work of high quality is to be undertaken, the measurements should be made

[8] Wilson, R. M., Triangulation Adjustments in Plane Coordinate Systems: *Engineering News-Record*, October 3, 1935.

directly from the negative, or at least from a glass diapositive, in order to avoid the distortion of a paper print.

The formulas for obtaining the angles from coordinates measured on the photograph are given on the specimen computing form, entitled "Oblique Photographs." In general, the same symbols are used on this form that have been employed in the discussion of the graphical method. The horizon line and the line perpendicular thereto through the center of the photograph, are the axes of the system in which the coordinates x and y are measured on the photograph. The greater proportion of sights from airplane camera stations are down from the horizon; this form was arranged, therefore, to treat y, V and tan V as positive when measured down, in order to avoid too frequent negative signs. These quantities have thus been reversed in sign for convenient use in this computation, but otherwise the symbols are consistent with those used in the graphical method. Each form provides space to compute 12 sights.

The three methods described here are essentially the same in principle. In using any one of them an estimated horizon line must be drawn as a beginning, and subsequently corrected to the true horizon. It may be convenient to use the graphical method as a preliminary to either of the other two, so that a horizon line that is very nearly correct may be drawn before the photograph is placed in either the photoalidade or the comparator.

Adjustment of Reference Plane to Many Observations

Three ground control points, suitably situated, are theoretically sufficient to locate and orient the camera station. But in using only three points there is no check against errors in identifying them on the photograph, or in the subsequent graphical construction and computation. It is desirable, therefore, to use more than three control points. But absolutely perfect observations cannot be expected, and if different three-point combinations are used in several different solutions, their several results will all be slightly different. The average of these might give a reasonably accurate general result, but to carry through the several similar solutions required would be tedious.

An adjustment by the method of least squares, to include impartially the influence of more than three control points in a simultaneous solution, is not difficult. It has been shown already how each observation upon a control point provides a set of coordinates, Z', L, and R, that can be substituted numerically into the general equation of a plane. This gives an equation with three quantities, a, b, and Z, still unknown, but with numerical coefficients which represent the observation made upon that control point. When a, b, and Z have been found by solving three such equations, they are written back into the general equation, thereby defining the reference plane. But if there are more than three such observation equations, they cannot all be satisfied exactly by any single set of values for a, b, and Z. Each observation equation, to be mathematically complete, should therefore include a term, r, to represent the amount by which that equation is not satisfied. Thus:

$$Z + aL_1 + bR_1 - Z_1' = r_1$$

$$Z + aL_2 + bR_2 - Z_2' = r_2$$

etc.

The residuals, r_1, r_2, etc., are analogous to the "deviations from the mean" that appear when each one of a number of separate measurements of any single quantity is individually compared with the mean of them all. If the basic data,

UNITED STATES
DEPARTMENT OF THE INTERIOR
GEOLOGICAL SURVEY

State______________

Quadrangle____________

OBLIQUE PHOTOGRAPHS

Computation of Horizontal and Vertical Angles

$c^2=(f')^2+d^2$; $\tan O=\frac{c\,x}{c^2-d\,y}$; $\tan V=\frac{f'y\cos O}{c^2-d\,y}$; $Z'=M\tan V+E-$(Curv.& Refr.)

Photo No.______ f'_____ d____ c²_______ c_____ Horizon______

1	Point						
2 (Photo)	x						
3 (Photo)	y						
4	d·y						
5	$c^2-d\cdot y$						
6	$\tan O=\frac{c\cdot x}{[5]}$						
7	sin O						
8	cos O						
9	$\tan V=\frac{f'\cdot y}{[5]}\cdot[8]$						
10 (Ground)	M						
11 (Ground)	E						
12	M·tan V						
13	Curv. & Refr.						
14	$Z'=[11]+[12]-[13]$						
1	Point						
2 (Photo)	x						
3 (Photo)	y						
4	d·y						
5	$c^2-d\cdot y$						
6	$\tan O=\frac{c\cdot x}{[5]}$						
7	sin O						
8	cos O						
9	$\tan V=\frac{f'\cdot y}{[5]}\cdot[8]$						
10 (Ground)	M						
11 (Ground)	E						
12	M·tan V						
13	Curv. & Refr.						
14	$Z'=[11]+[12]-[13]$						

f' = focal length of camera reduced to scale of photograph, in inches.
d = distance from center of photograph to assumed horizon, in inches.
x = distance to right of axis of photograph, in inches.
y = distance below assumed horizon, in inches.
O = horizontal angle at camera.
V = vertical angle at camera.
M = ground distance from camera station, in feet.
E = ground elevation, in feet.
Z' = elevation of projection of point in plane of camera and assumed horizon, in feet.

Photo measured by______________
Date______________
Map measured by______________
Date______________
Computed by______________
Date______________

the computations, and the observations were all absolutely perfect, the residuals would all be zero; but of course this is never the case in practice. The least squares solution obtains the set of values for a, b, and Z that makes the sum of the squares of the residuals as small as possible; in other words, it defines the plane that most nearly fits all the observations.

The numerical example of an algebraic solution, using only three control points, provided corrections applicable to the tentative reference plane. A new horizon line, B–B, Figure 2, was drawn on the photograph to conform as nearly as possible to the indicated corrections. The following table contains the data for observations made upon the original 3 control points and 10 more, making 13 in all, which are referred to the corrected horizon line. In this table the values of $\sqrt{L^2+R^2}$ are slightly larger than M; the scaled values of M were reduced by proportional corrections because the scale of the map from which they were measured was not quite correct. The residuals in the right hand column are obtained after the equations are solved.

Solution by Least Squares

To apply least squares to these data, the 13 observation equations might be written out in full, as was done for the previous algebraic solution. But there are contained in the table all essential figures for each observation equation, so that the usual normal equations can be formed directly from the table. The coefficient of each Z term is equal to 1; the coefficients of a and b are listed under the headings L and R, respectively. The absolute term is found in the Z' column, but it

Referred to Corrected Horizon Line $B-B$

$f'=11.583$ $d=1.505$ $c=11.680$
$\tan i=-.12993$ $i=-7°24'10''$ $\cos i=.99166$

Point	Feet ÷ 1,000 Scaled from Tracing			Inches Scaled from		Feet ÷ 1,000			Ft.
	L	R	M	Photo y	Tracing m	$\frac{My \cos i}{m}$	$E'=E-$Curv. & Ref.	Z'	r
1	25.18	−3.92	25.43	+0.483	11.875	+1.026	3.114−.013	2.075	−19
2	9.56	−3.07	10.02	−1.065	12.140	− .872	1.194−.002	2.064	− 6
3	17.62	+5.92	18.55	+0.047	12.325	+ .070	2.128−.007	2.051	− 1
4	19.89	−1.66	19.92	+0.354	11.755	+ .595	2.652−.008	2.049	+ 6
5	20.00	−5.16	20.62	+0.080	12.065	+ .136	2.185−.009	2.040	+17
6	34.29	−2.66	34.32	+0.728	11.805	+2.099	4.197−.024	2.074	−21
7	8.33	−2.86	8.79	−1.260	12.200	− .900	1.159−.002	2.057	+ 1
8	25.04	+3.06	25.18	+1.121	11.910	+2.350	4.405−.013	2.042	+ 9
9	24.98	+5.50	25.53	+1.112	12.095	+2.328	4.380−.013	2.039	+10
10	18.29	+0.11	18.25	+0.090	11.690	+ .139	2.200−.010	2.051	+ 3
11	8.68	+2.82	9.11	−1.206	12.115	− .899	1.164−.002	2.061	− 7
12	28.85	−2.90	28.94	+0.472	11.795	+1.148	3.223−.017	2.058	− 4
13	33.83	−8.45	34.80	+0.818	12.135	+2.326	4.395−.025	2.044	+14

By least squares solution:
$a=-.00016$ $b=-.00071$ $Z=2.0569$

must be given a negative sign to transpose it to the same side of the equation with the other terms. Let "Sum" mean the sum of the indicated products taken successively from the lines 1 to 13 inclusive. Using such sums, the terms of the three normal equations (omitting the terms to the left of the diagonal terms as is customary) are represented thus:

Z	a	b	Z' (absolute)
$+\text{Sum}(1^2)$	$+\text{Sum}(1\times L)$	$+\text{Sum}(1\times R)$	$-\text{Sum}(1\times Z')$
	$+\text{Sum}(L^2)$	$+\text{Sum}(L\times R)$	$-\text{Sum}(L\times Z')$
		$+\text{Sum}(R^2)$	$-\text{Sum}(R\times Z')$

The detailed theory for forming and solving such equations[8,9] is given in textbooks on least squares, but brief practical directions to continue the solution are given here in the form of a diagrammatic "Key to Routine Solution." To avoid large figures it is permissible to subtract arbitrarily some convenient amount from all values of Z'. In the present example they have been diminished by 2.000 (two thousand feet), which has the effect of referring the solution to a plane at that distance above datum; when the solution is finished this amount can be replaced. Meanwhile, there will be only two significant figures for Z', instead of four, which will simplify the solution. With this modification, the numerical substitution in this example produces these terms:

Z	a	b	$Z'-2.000$ (absolute)
+13.	+ 274.54	− 13.27	− 0.705
	+6714.36	−404.02	−14.836
		+231.86	+ 0.856

Still another modification may be employed to make the solution easier. The coefficients for a and b in these normal equations are large; to make them more consistent with the magnitude of the Z coefficient it is better to solve for $10a$ and $10b$, which justifies shifting the decimal point 2 places to the left in (L^2), $(L\times R)$, and (R^2), and one place to the left in $(1\times L)$, $(1\times R)$, $(L\times Z')$, and $(R\times Z')$.

Solution of Normal Equations

The modified normal equations appear in the table entitled "Solution of Normal Equations." The column, S, to the right of the "Absolute" column, contains the check terms usually carried through such computations to guard against errors.

Perhaps the simplest way to show the systematic routine of the solution is by a diagrammatic "key." Each line in the solution in this example has been given a designating number to use, with the letters in the column headings, for reference purposes. Thus, $b2$ means the figure appearing in column b, line 2. The figures found for the coefficients of the terms in the normal equations are written as indicated in lines 1, 3, and 7 in both of the following tables.

Referring now to the "Key to Routine Solution," the first step is to add selected given figures, as indicated (remembering to change the sign of the sum) to obtain the check figures for the S column, lines 1, 3, and 7. When that has been done, the solution progresses as indicated, line by line, to the bottom of the form. The figures found in each line marked by an asterisk should add nearly to zero as a check on the computation. Lines 2, 6, and 11 will each begin with −1; this should not be forgotten when adding along these lines to obtain the check. The "key" provides a guide that will not be needed after the first ex-

[9] Norris, Clarence and Speert, Julius L., An Improved Method for Adjusting Level and Traverse Surveys: *Am. Soc. C. E. Transactions* 1940, page 1376, 1940 Paper No. 2087, see appendix II.

KEY TO ROUTINE SOLUTION

Ref. No.	Z	$10\times a$	$10\times b$	$Z'-2.000$ (Absolute)	S (Sum)
1	$+13.0000$	$+27.4540$	-1.3270	$-\ .7050$	($-$Sum: Z to Z' line 1)
2		(Z to S inclusive, line 1, divided by $-Z1$)*			
3		$+67.1436$	-4.0402	-1.4836	($-$Sum: $a1$, $a3$, $b3$, $Z'3$)
4			(a to S, incl., line 1, multiplied by $a2$)		
5			(Sum, lines 3 and 4)*		
6			(a to S, incl., line 5, divided by $-a5$)*		
7			$+2.3186$	$+0.0856$	($-$Sum: $b1$, $b3$, $b7$, $Z'7$)
8			(b to S, incl., line 1, multiplied by $b2$)		
9			(b to S, incl., line 5, multiplied by $b6$)		
10			(Sum, lines 7, 8, and 9)*		
11			(b to S, incl., line 10, divided by $-b10$)*		
12			(Copy $Z'11$)	$=10\times b$	
13		$b12\times b6$, $+Z'6$		$=10\times a$	
14	($a13\times a2$ $+b12\times b2$ $+Z'2$)			$=Z-2.000$	

SOLUTION OF NORMAL EQUATIONS

Ref. No.	Z	$10\times a$	$10\times b$	$Z'-2.000$ (Absolute)	S (Sum)
1	$+13.0000$	$+27.4540$	-1.3270	$-\ .7050$	-38.4220
2	$-\ 1.$	$-\ 2.1118$	$+\ .1021$	$+\ .0542$	$+\ 2.9555$
3		$+67.1436$	-4.0402	-1.4836	-89.0738
4		-57.9774	$+2.8024$	$+1.4888$	$+81.1396$
5		$+\ 9.1662$	-1.2378	$+\ .0052$	$-\ 7.9342$
6		$-\ 1.$	$+\ .1350$	$-\ .0006$	$+\ .8656$
7			$+2.3186$	$+\ .0856$	$+\ 2.9630$
8			$-\ .1355$	$-\ .0720$	$-\ 3.9229$
9			$-\ .1671$	$+\ .0007$	$-\ 1.0711$
10			$+2.0160$	$+\ .0143$	$-\ 2.0310$
11			$-1.$	$-\ .0071$	$+\ 1.0074$
12			$-\ .0071$	$=10\times b$	
13		$-\ \ .0016$		$=10\times a$	
14	$+\ \ .0569$			$=Z-2.000$	

perience, when the computer should discover and remember the systematic sequence in the line-by-line progress of such solutions. With a computing machine or a slide rule the solution of the three normal equations may be done in less than an hour.

The immediate results of this solution give figures for $10\times a$, and $10\times b$, and $Z-2.000$ because of the modified normal equations used. Therefore the first two must be divided by 10, and 2.000 must be added to the third to obtain the results sought. In this example, then, $a=-.00016$, $b=-.00071$, $Z=2.057$; the latter is the elevation of the camera station in thousand-foot units, and agrees well with the figure of elevation found in the algebraic solution using three control points.

The values of a and b indicate that a little too much correction was applied to the first estimation, and the reference plane now has a small slope in the opposite direction. But, instead of trying to draw another horizon line, it is better to preserve the new reference plane, and with the accurate knowledge of its slight slope, make appropriate corrections to any observations referred to it.

The elevation of a point on the adopted reference plane, situated just above a ground position whose horizontal coordinates are L and R, is equal to $Z+aL+bR$, as referred to the datum plane tangent to the earth's surface at N. In the example being discussed, this elevation is:

$$2.057 - .00016\,L - .00071\,R.$$

The figure so found should be used with the vertical angle observation to transfer an accurate elevation to a ground point in that position. In ordinary ground surveying the analogous procedure uses the elevation of the station itself, because the vertical angle is then controlled by a spirit level, and so it is taken for granted that the reference plane is horizontal.

Residuals

The value of Z and the slope of the reference plane were determined by combining or averaging the observations upon 13 different ground points. To see how nearly each individual observation is satisfied by the adopted reference plane, the elevation on the plane may be compared with the elevation indicated by that particular observation. It must be remembered here that elevations are being referred to the datum plane, not to the curved surface of the earth beneath it. In the present example, the elevation of the reference plane over point No. 9 is:

$$2.057 + (-.00016)(+24.98) + (-.00071)(+5.50) = 2.049.$$

The value for Z' indicated by the observation is given in the table as 2.039, so this particular observation deviates from the combined result of all thirteen by 10 feet. However, if the slope of the reference plane were not considered, the deviation obtained by comparing this observation directly with the elevation of the camera station would be 18 feet. What has just been done is the substitution of known figures into the observation equation:

$$Z + aL + bR - Z' = r.$$

For point No. 9, therefore, $r=+10$. By similar substitutions made into the observation equations for other points, always using $Z=2.057$, $a=-.00016$, and $b=-.00071$ to represent the adopted plane, the other residuals are obtained. They are all given in the right hand column of the table of observations.

Magnitude of Residuals as a Test of Map Accuracy

The example given is based on the graphical method alone. The 13 points used to control the solution were not established particularly for that purpose. They were merely identifiable features in the field of view, selected from a topographic map published on the scale of 1:24,000. The elevations of some of them were estimated by interpolating between 10-foot contours.

Residuals are caused by a combination of various accidental errors, such as the adoption of inaccurate position or elevation for the ground points or errors in the map, identifying the ground points imperfectly on the photograph, distortion of the print or in the camera lens, inaccurate graphical construction on the tracing paper.

The average magnitude of the 13 residuals just listed is 9 feet, and is a numerical index of the precision of the data used as well as of the graphical method itself.

Accurately plotted positions and indication of correct elevation are qualities that make a good topographic map. If details have been correctly mapped, there will appear only very small relative discrepancies when observations to them are compared simultaneously through a least squares solution. Even if the points are not formal control points, the photograph will still provide information as to the accuracy of their relative elevations. If equivalent precision in method and similar photographs are used in different areas covered by maps of various degrees of accuracy, it is reasonable to expect that the magnitude of the residuals will vary according to map accuracy. The adoption of one of the more nearly accurate methods as standard procedure would soon show what magnitude to expect in the residuals corresponding to each designated grade of map. Considered thus, one of the methods described may so be used to test the accuracy of elevations on any map, regarding the average residual in each test as the numerical index of the quality of the map tested. The scheme is at the same time effective in testing relative accuracy of plotted horizontal positions, because errors in position will produce discrepancies of elevation as observed by the vertical angles and scaled distances. Also, if appreciable errors in horizontal position exist in the map, they will be disclosed when resecting for the camera station.

Interpolation of Additional Control

Oblique photographs include large fields of view as compared to the usual vertical photographs. They show near and distant control points in proper relation to foreground detail; this quality makes them adaptable to the interpolation of supplementary control, both horizontal and vertical, to aid many of the established procedures of surveying and photogrammetry. Slotted templets[10] may be made using horizontal directions extracted from oblique photographs; these may be included in an adjustment of vertical photograph templets to tie in the whole group with distant control.

In conclusion the author wishes to offer his opinion that many promising potential uses for oblique photographs are now suffering undeserved neglect. It is hoped that the principles[11,12] described here may help to promote their use.

[10] Kelsh, Harry T., The Slotted-Templet Method for Controlling Maps made from Aerial Photographs: U. S. Department of Agriculture, *Miscellaneous Publication No. 404*, November 1940.

[11] Wang, Dr. Chih-cho, China Institute of Geography, Height Determinations with High Oblique Photographs: *Photogrammetric Engineering*, Vol. VII, No. 3, July-August-September 1941.

[12] See also references listed in PHOTOGRAMMETRIC ENGINEERING, April-May-June issue, 1938, page 74, under the article (2) above.