ANALYTICAL AND GRAPHICAL RECTIFICATION OF A TILTED PHOTOGRAPH

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INTRODUCTION

AN AERIAL photograph fails being a perfect map of the ground photographed because of unavoidable tilt in the photograph and because of the presence of relief on the ground. To convert the photograph into a true map of the ground, then, it must be corrected for both tilt and relief. This correction program can be performed in two steps, the first step correcting the photograph for map displacements induced by tilt, and the second step correcting that result for map displacements induced by relief. The result of the first step is to produce an intermediate map having the appearance of what the photograph of

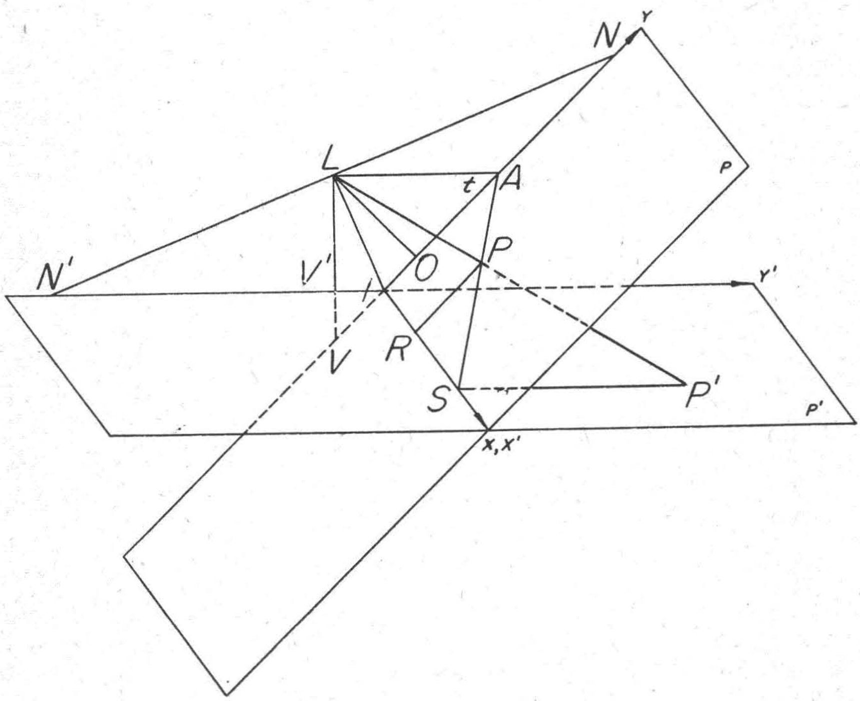

FIGURE 1

the ground would have been like had there been no tilt at the time of exposure, the exposure station being at exactly the same point in space as before. We call this intermediate map the *rectification* of the photograph. The following is believed to be a new analytical and graphical treatment of this first, and more difficult, part of the correction program. We assume that the tilt and the swing of the photograph have already been computed by any of the present acceptable methods and proceed with the problem from that point.

Analytical Rectification of Photograph

Let p denote the plane of the tilted photograph, O the principal point of p, I the isocenter (hereafter referred to as the *positive isocenter*) of p, V the nadir point of p, and L the exposure station. The line in p through I and perpendicular to the principal line VO is called the *tilt axis* of p, and is horizontal in space. Let p' be the horizontal plane passing through the tilt axis of p. The angle between planes p and p' is the tilt, t, of plane p. Let us, from L as vertex, project plane p upon plane p'. The resulting map on p' is, then, the rectification of the photograph p. It is the purpose of this section to develop formulas for this rectification of p upon p'.

To such an end take I on p and p' as origins of rectangular coordinates, IO and the tilt axis as y- and x-axes respectively on p, the projection of IO and the tilt axis as y'- and x'-axes respectively on p'. Let $P:(x, y)$ be any point on p and let $P':(x', y')$ be the projection of P on β'. In plane LIO, which is perpendicular to the tilt axis IS (see Fig. 1), draw NLN' perpendicular to LI to cut the y-axis in N and the y'-axis in N', and draw LA parallel to $N'I$ to cut IN in A. Since, by definition of I, LI bisects the angle between LO and the vertical LV, it easily follows that $IN=IN'$ and $LA=IA=\frac{1}{2}IN$. Let AP cut the tilt axis in S, and let R be the foot of the perpendicular from P on the tilt axis. Since LA is parallel to SP' it follows that SP' is perpendicular to IS. Designate the length $LA=IA$ by a. Then

$$SP'/LA = PS/AP = RS/IR.$$

That is

$$y'/a = (x' - x)/x. \tag{1}$$

Similarly

$$RP/IA = PS/AS = SP'/(SP' + LA).$$

That is

$$y/a = y'/(y' + a). \tag{2}$$

Solving (1) and (2) for x' and y' in terms of x and y we find

$$x' = ax/(a - y), \qquad y' = ay/(a - y). \tag{3}$$

Now, designating the focal length LO by f, we see, from the figure, that

$$a = LA = LO/\sin(LAI) = f/\sin t. \tag{4}$$

Therefore, the coordinates of P' are given in terms of the coordinates of P by relations (3), where a is determined by (4). These relations constitute analytical formulas for the rectification of photograph p. We rewrite them here for future reference—

$$x' = ax/(a - y), \qquad y' = ay/(a - y), \qquad a = f/\sin t.^* \tag{5}$$

Two Consequences of the Rectification Formulas

From (5) we observe that

$$x'/y' = x/y.$$

* In a future paper the author hopes to apply these formulas to an analytical discussion of the curves appearing on parallax correction graphs used in stereo-comparagraph and contour-finder technique.

Therefore

$$\text{angle } yIP = \text{angle } y'IP'. \tag{6}$$

This preservation of angles at I is a well-known property of the positive isocenter.

We shall refer to the point N (for a reason soon to become apparent) as the *negative isocenter* of p, and its projection, N', as the negative isocenter of p'. Now $IN' = IN = 2a$, and we have, from (5),

$$\frac{x'}{2a + y'} = \frac{ax/(a - y)}{2a + ay/(a - y)} = \frac{x}{2a - y}\,.$$

Therefore

$$\text{angle } INP = \text{angle } IN'P'. \tag{7}$$

Thus angles at the negative isocenter N are also preserved under the projection. However, if we regard as the positive sides of p and p' those sides visible from L, then we see that, under the projection, angles at I are preserved in both magnitude and sign, but angles at N are preserved in magnitude only and change in sign. This is the reason for the names *positive* and *negative* isocenters for I and N respectively.

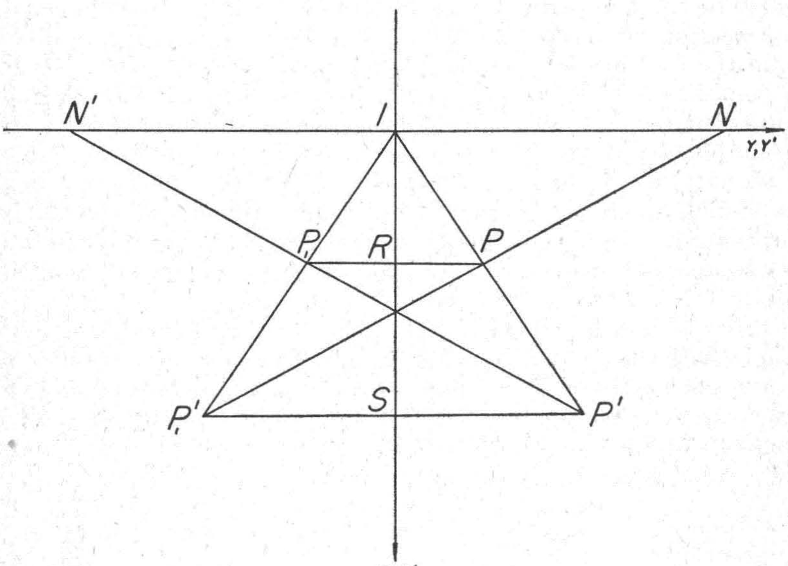

FIGURE 2

GRAPHICAL RECTIFICATION OF PHOTOGRAPH

Relations (6) and (7) of the last section furnish a theoretically elegant graphical solution for the rectification of a photograph. Let us superimpose plane p' on plane p so that the x'- and y'-axes of p' fall, respectively, on the x- and y-axes of p. The construction for finding P' from P is, then (see Fig. 2): *Reflect P in the tilt axis IS to obtain P_1; draw IP and $N'P_1$ to meet in P'.* The photographic image P is in this manner displaced to P', the position where it would have been had there been no tilt in the photograph.

If a series of photographs of a flight strip are rectified in this manner, then the second part of the correction program is readily carried out by the method of radial plotting. (See, e.g., Chapter IV of Church's *Elements of Aerial Photogrammetry.*)

REMARKS CONCERNING ALMOST VERTICAL PHOTOGRAPHS AND HIGH OBLIQUE PHOTOGRAPHS

In the case of small tilts the above graphical solution runs into some drafting difficulties. For, if t is very small, then $IN' = 2f/\sin t$ is very large. In such cases the point N' itself cannot be plotted. Here it is perhaps best to define N' graphically as the inaccessible point of intersection of two lines. To do this, select any convenient point P not near IS (see Fig. 2) and then plot P' by means of the rectification formulas (5). Then find P_1, the reflection of P in IS. $P'P_1$ now passes through N'. A line through N' and any other point Q of p may now be found by one of the drafting tricks devised for this purpose.

A combination of analytical and graphical methods may be used. For example, if two points P and Q have equal y ordinates, then P' and Q' also have equal y' ordinates. We might, then, by means of the second of the formulas (5), compute, for a number of different values of y, the y ordinate corrections to be applied. If such a table of corrections were made for a suitably small increase in y, interpolation could be used, and any line parallel to IS would shift to another line parallel to IS. The points themselves would then be located on the shifted line by projecting rays from I.

It is in the case of high oblique photographs, however, that the graphical solution would seem to be of particular value. For here N' will fall well within reasonable distance of I and no drafting tricks will be needed. As a matter of fact the author has devised for this case a rather obvious linkage motion for quickly tracing the rectified map from the photograph. Then, too, in the case of high obliques (where the horizon line is visible), the preliminary analytical or graphical computation of the tilt and swing of the photograph is extremely simple when compared with the computation for these things in a photograph of small tilt.

All in all, one wonders if this graphical method, performed on high oblique photographs with the aid of a good mechanical linkage motion, would not be suprior to the perspective grid method used so much by the Canadian Government. For the new method appears not only to be fast, but has the decided advantage of working equally well on either flat or rugged terrain. The perspective grid method is good only for flat ground.