## A DERIVATION FOR THE IMAGE DISPLACEMENT DUE TO TILT*

Michael G. Misulia, Photogrammetrist<br>U. S. Coast and Geodetic Survey

SEVERAL different equations for the displacement of an image on a tilted photograph have been derived by various authors in the past. This displacement has been computed as: $:^{1,2,3}$

$$
\begin{equation*}
d=f\left[\tan (\alpha+t)-\tan \alpha-2 \tan \frac{1}{2} t\right] \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
d^{\prime}=f\left[\tan \beta-\tan (\beta-t)-2 \tan \frac{1}{2} t\right] . \tag{2}
\end{equation*}
$$

In somewhat simpler equations the image displacement has also been computed as : $^{4,5,6}$

$$
\begin{align*}
& d=\frac{(i a)^{2}}{\frac{f}{\sin t}-i a}  \tag{3}\\
& d^{\prime}=\frac{(i b)^{2}}{\frac{f}{\sin t}+i b} . \tag{4}
\end{align*}
$$

Where,
$d=$ the displacement for an image, parallel to the principal line and due to tilt, on the upper side of a photograph
$d^{\prime}=$ the displacement for an image, parallel to the principal line and due to tilt, on the lower side of a photograph.
All of the other symbols used in the above equations are illustrated in the figure.

Equations (3) and (4) are equivalent to equations (1) and (2) respectively.
The purpose of this article is to show a more direct and shorter method than has been used by Messrs. R. O. Anderson and J. L. Rihn for the derivation of this image displacement due to tilt. The derivation offered makes use of proportional triangles and the application of some basic principles in geometry and trigonometry.

Considering the isocenter, $i$, as the center of rotation, swing an arc from point $a$ on the tilted photograph so as to intersect the equivalent vertical photograph at a point $c$. The distances $i a$ and $i c$ are therefore equal and the isosceles triangle iac is thereby formed having two sides equal to $+r$.

[^0]

Figure 1
Symbols used in the figure are defined as follows:
$t=$ angle of tilt
$p=$ principal point
$i=$ isocenter
$n=$ nadir point
$o=$ perspective center (lens)
$f=$ focal length of photograph
$\alpha=\angle p O a=$ angle at the perspective center $O$ measured counter-clockwise from the photograph perpendicular to any point $a$ on the upper side of the photograph
$\beta=\angle p O b=$ angle at the perspective center $O$ measured clockwise from the photograph perpendicular to any point $b$ on the lower side of the photograph
$a, b=$ images of ground points in the principal plane of the photograph
$a^{\prime}, b^{\prime}=$ corresponding points on the equivalent vertical photograph
$+r=$ distance $i a=$ distance $i c$
$-r=$ distance $i b=$ distance $i e$
$d=$ the displacement for an image, parallel to the principal line and due to tilt, on the upper side of a photograph
$d^{\prime}=$ the displacement for an image, parallel to the principal line and due to tilt, on the lower side of a photograph

$$
\angle O i p=90^{\circ}-\frac{t}{2}
$$

In the isosceles triangle iac,

$$
\angle i a c=90^{\circ}-\frac{t}{2}
$$

Therefore lines $O i$ and $a c$ are parallel (corresponding opposite interior angles)
and,

$$
\triangle O i a^{\prime} \text { is therefore similar to } \triangle a c a^{\prime} .
$$

By proportion,

$$
\frac{d}{r+d}=\frac{a c}{O i}
$$

However,

$$
\begin{aligned}
& a c=2 r \sin \frac{1}{2} t \\
& O i=\frac{f}{\cos \frac{1}{2} t}
\end{aligned}
$$

Substituting,

$$
\begin{aligned}
\frac{d}{r+d} & =\frac{2 r \sin \frac{1}{2} t}{\frac{f}{\cos \frac{1}{2} t}} \\
d & =\frac{r 2 \sin \frac{1}{2} t \cos \frac{1}{2} t}{f}(r+d)
\end{aligned}
$$

By trigonometry,

$$
2 \sin \frac{1}{2} t \cos \frac{1}{2} t=\sin t
$$

Therefore,

$$
\begin{gathered}
d=\frac{r \sin t(r+d)}{f} \\
d f=r^{2} \sin t+d r \sin t \\
d(f-r \sin t)=r^{2} \sin t \\
d=\frac{r^{2} \sin t}{f-r \sin t}
\end{gathered}
$$

Dividing the numerator and the denominator by $\sin t$,

$$
\begin{equation*}
d=\frac{r^{2}}{\frac{f}{\sin t}-r} \tag{5}
\end{equation*}
$$

This form of the equation applies to both sides of the photograph when the algebraic sign of $r$ is observed. The equation is exactly the same as equations (3) and (4), the values of $i a$ and $i b$ being equal to $+r$ and $-r$ respectively.

Equation (5) is usually expressed in an approximate form when computing the value of $d$ for either side of a single lens photograph having a tilt angle of less than $3^{\circ}$.

$$
\begin{equation*}
d=\frac{r^{2} \sin t}{f} \tag{6}
\end{equation*}
$$

Equation (6) is logical due to the fact that the subtractive value of $r$ in the denominator of equation (5) has only a small effect on the value of $d$ because of the much larger values of $r^{2}$ and $f / \sin t$.


[^0]:    * Published by permission of The Director, U. S. Coast and Geodetic Survey.
    ${ }^{1}$ Bagley, James W., Aerophotography and Aerosurveying, McGraw-Hill Book Company, Inc., New York and London, 1941, Chapter VII.
    ${ }^{2}$ Sharp, H. Oakley, Photogrammetry, 3rd Edition, John Wiley and Sons, 1943.
    ${ }^{3}$ Breed and Hosmer, Higher Surveying, 5th Edition, John Wiley and Sons, 1940, pp. 392-397..
    ${ }^{4}$ Anderson, Ralph, O., Applied Photogrammetry, Edwards Brothers, Inc., Ann Arbor, Michigan, 1939, pp. 90-92.
    ${ }^{5}$ Rihn, Jack L., Manual of Photogrammetry, preliminary edition, 1944, pp. 281-282.
    ${ }^{6}$ Tewinkel, G. C., Manual of Photogrammetry, preliminary edition, 1944, pp. 266-267.

