A PROOF OF THE ISOCENTER PRINCIPLE

Herman J. Shea, Assistant Professor of Surveying,
Massachusetts Institute of Technology

VARIOUS proofs establishing the all-important property that displacements caused by tilt radiate from the isocenter have been offered from time to time. Some of the proofs have been based on logic and inference while others have the property of being more or less mathematically exact. The proof offered in this article, it is believed, is mathematically exact and is based on moderately simple analytical geometry and spherical trigonometry.

In the tilted photograph, a rectangular coordinate system is set up using the principal point as the origin, the X-axis along the principal line, and the Y-axis parallel to the axis of tilt. The positive end of the X-axis is taken in the direction of the raised portion of the tilted print. A point, a, having coordinates (x, y) is assumed to lie in the first quadrant (Fig. 1).

The angle α is taken as the angle at o between the principal distance and the perspective ray through point a. From Figures 1 and 2, it will be seen that the following relations for α may be obtained easily:

$$\tan \alpha = \frac{\sqrt{x^2 + y^2}}{f}; \quad \cos \alpha = \frac{f}{\sqrt{x^2 + y^2 + f^2}}; \quad \sin \alpha = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + f^2}}. \tag{1}$$

In the plane of the tilted photograph, the angle ϕ is the angle at the principal point between the positive end of the X-axis and the point a. The following relations for ϕ are quite evident:

$$\tan \phi = \frac{y}{x}; \quad \sin \phi = \frac{y}{\sqrt{x^2 + y^2}}; \quad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}. \tag{2}$$

The next step is to set up a spherical triangle with o as the center of the sphere involving the ray through a, oi, and the principal distance, op. For purposes of identification, the spherical triangle is lettered $a'p'i'$ (Fig. 3). It will be noticed that the sides $a'p'$ and $i'p'$ are equal respectively to α and $t/2$ and that the angle $a'p'i'$ equals $180° - \phi$. Two other parts of this triangle are of interest in this proof. The side $i'p'(\beta)$ represents the angle at o between the perspective rays through the iso-center and a. The angle at $i'(\theta)$ is the dihedral angle along the edge oi' between the plane $oi'a'$ and the principal plane. It is important to note that this angle, when referred to i, will lie in a plane half-way between the planes of the tilted photograph and the equivalent vertical photograph. It is therefore neither a measure of the true horizontal angle nor of the angle in the plane of the tilted photograph. Later a relation will be derived by which the true horizontal angle (θ_h) corresponding to the dihedral angle θ may be found.

By use of the cosine law of spherical trigonometry

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{3}$$

the following relation for β may be stated:

$$\cos \beta = \cos \alpha \cos \frac{t}{2} - \sin \alpha \sin \frac{t}{2} \cos \phi. \tag{4}$$

Having β it is now possible to find θ through use of the sine law of spherical

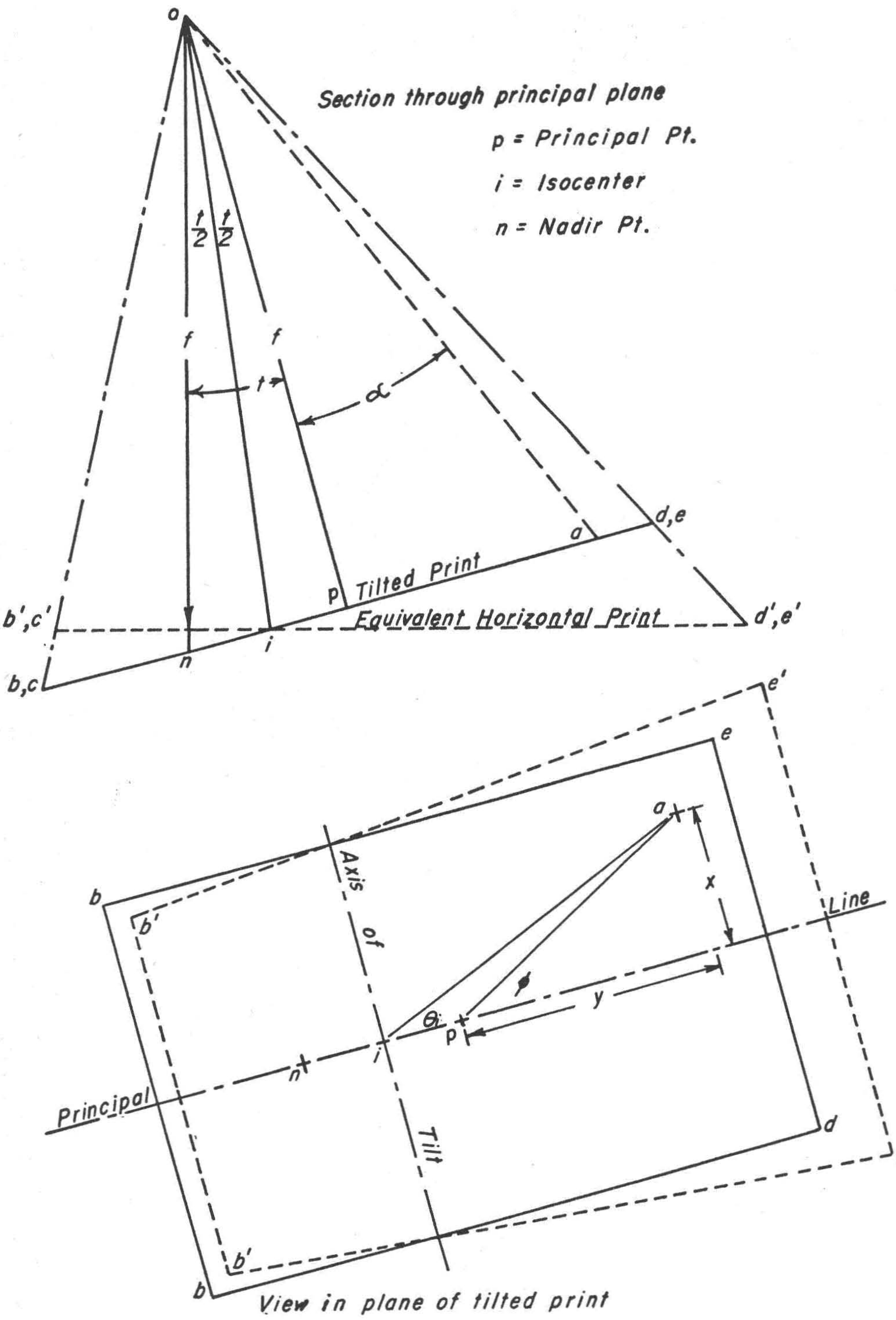
o
Section through principal plane
p = Principal Pt.
i = Isocenter
n = Nadir Pt.
t/2 t/2
f
f
t
α
a
d,e
p Tilted Print
b',c'
Equivalent Horizontal Print
d',e'
n
i
b,c
e'
e
a
Axis of Tilt
x
Line
b
b'
φ
y
θ
p
i
n
Principal
d
d'
b'
b
View in plane of tilted print

FIG. 1.

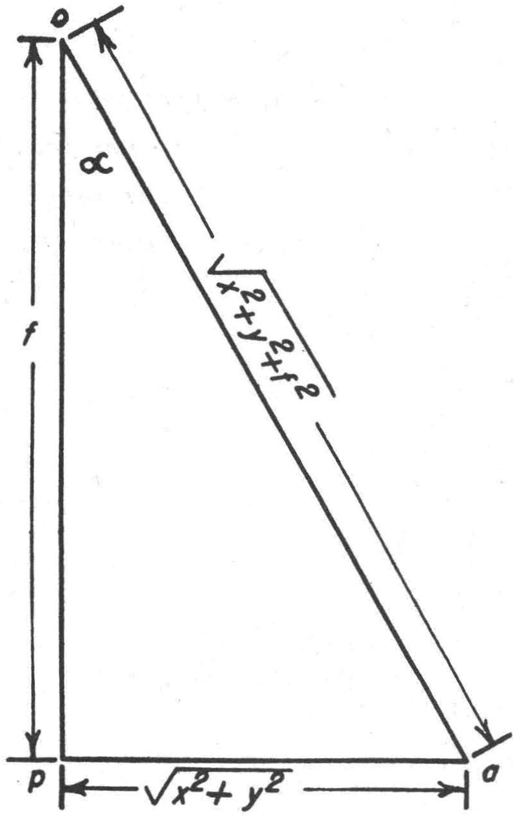

FIG. 2.

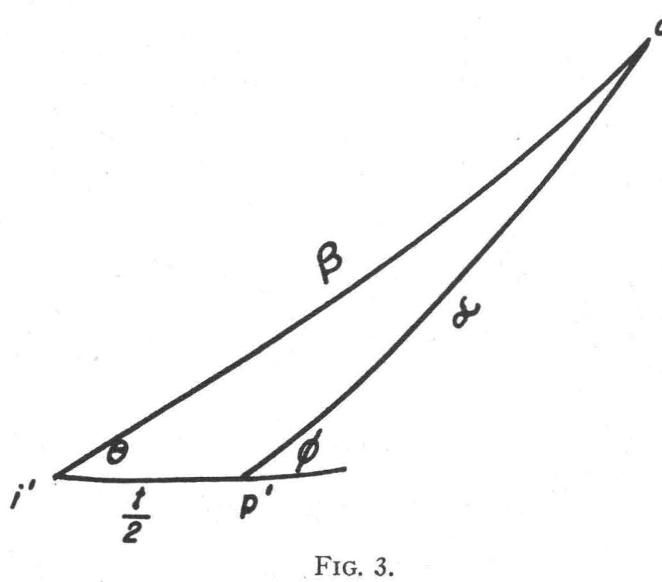

FIG. 3.

trigonometry

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} \tag{5}$$

or

$$\sin \theta = \sin \alpha \frac{\sin \phi}{\sin \beta} \cdot \tag{6}$$

As stated before, the angle θ is a dihedral angle which, when referred to i, lies in a plane midway between the tilted print and the equivalent horizontal print. By reference to Figure 4, it will be noted that a spherical triangle with i as the

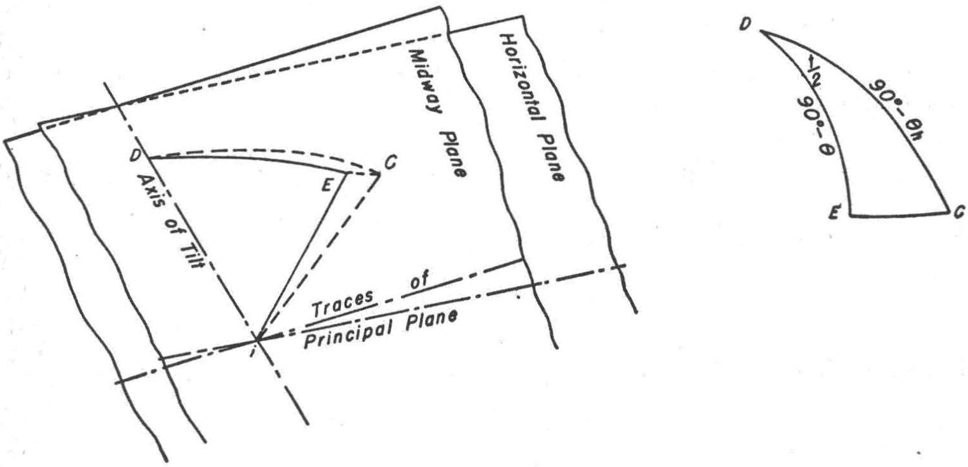

FIG. 4.

center of the sphere may be formed involving the axis of tilt, the plane of the equivalent horizontal photograph, and the plane containing θ. Point E lies in the midway plane and C in the plane of the equivalent horizontal photograph. This is a right spherical triangle with the right angle at E. From these data, the following relation may be derived:

$$\tan \theta_h = \tan \theta \cos \frac{t}{2} \cdot \tag{7}$$

The angle θ_h is worthy of especial attention, for this quantity represents the correct horizontal direction (referenced to the principal plane) of the point a.

There is one more quantity of interest which is best introduced at this point. The angle in the plane of the tilted photograph at the isocenter between the positive end of the X-axis and the point a is taken to be θ_i. The following relation for θ_i (Fig. 1) is readily evident:

$$\tan \theta_i = \frac{y}{(x + f \tan t/2)} \cdot \tag{8}$$

If, through the steps outlined above, it is possible to prove that θ_h and θ_i are equal, the principle that displacements caused by tilt radiate (in the plane of the tilted photograph) from the isocenter will be rigorously proven.

The proof is started through applying the relations stated in equations (1) and (2) to equation (4). This latter relation then becomes

$$\cos\beta = \frac{f}{\sqrt{x^2+y^2+f^2}}\cos\frac{t}{2} - \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+f^2}}\sin\frac{t}{2}\,\frac{x}{\sqrt{x^2+y^2}}$$
$$= \frac{(f\cos t/2 - x\sin t/2)}{\sqrt{x^2+y^2+f^2}}. \tag{9}$$

Through use of the identity that $\sin x = \sqrt{1-\cos^2 x}$, the following relation for $\sin\beta$ may be obtained

$$\sin\beta = \sqrt{\frac{(f\sin t/2 + \cos t/2)^2 + y}{x^2+y^2+f^2}}. \tag{10}$$

Substituting the relations (10), (1) and (2) into (6), it will be found that the following relation for θ will result:

$$\sin\theta = \frac{y}{\sqrt{(f\sin t/2 + x\cos t/2)^2 + y^2}}. \tag{11}$$

Through use of the identity,

$$\tan x = \sqrt{\frac{\sin^2 x}{1-\sin^2 x}},$$

it will be found that

$$\tan\theta = \frac{y}{(f\sin t/2 + x\cos t/2)}. \tag{12}$$

Applying this to equation (7)

$$\tan\theta_h = \tan\theta\cos\frac{t}{2} = \frac{y\cos t/2}{(f\sin t/2 + x\cos t/2)} = \frac{y}{(f\tan t/2 + x)}. \tag{13}$$

Comparing this last relation with that for $\tan\theta_i$ given in (8), it will be noticed that the two are quite identical thereby proving that displacements caused by tilt are indeed radial from the isocenter.