THE RECTIFICATION OF A TILTED AERIAL PHOTOGRAPH*

The following article attempts to present the problems involved in rectifying a tilted aerial photograph in such a way that a student will need only an elementary knowledge of photogrammetry, trigonometry, and geometry to understand the technique of rectifications and the computations necessary to established the settings of the rectifying camera.

THIS demonstration applies to a rectifier which has a fixed lens and a negative carrier so built that the tilt axis of the aerial negative can be swung parallel to the tilt axes of the negative carrier and easel. Furthermore, the negative carrier must permit a displacement of the negative in the direction of its principal line.

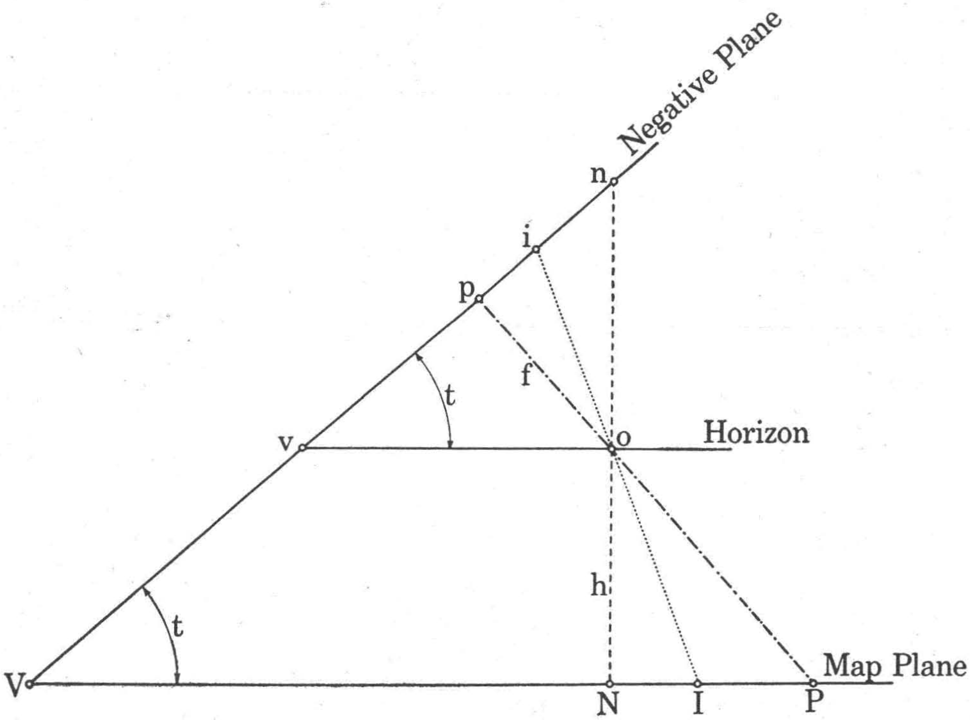

FIG. 1

Figure 1 shows the conditions under which an example aerial photograph was taken. The distance *po* represents the focal length (f) of the taking camera. The flying height at the scale of the map or at the scale of the rectified print is represented by h. The airtilt (t) is the angle between the negative and map planes measured in the principal plane. N and n, respectively, represent the nadir points on the map and on the photograph. P and p represent the principal points also on the map and on the photograph respectively; I and i are the isocenters.

Therefore if o is the lens point, the angle $NoP = pon = t$.

The point v represents the vanishing point on the photograph. This point lies in the horizontal plane which passes through the lens (o) of the taking camera. All lines parallel to the principal line on the map will meet at this vanishing point (v).

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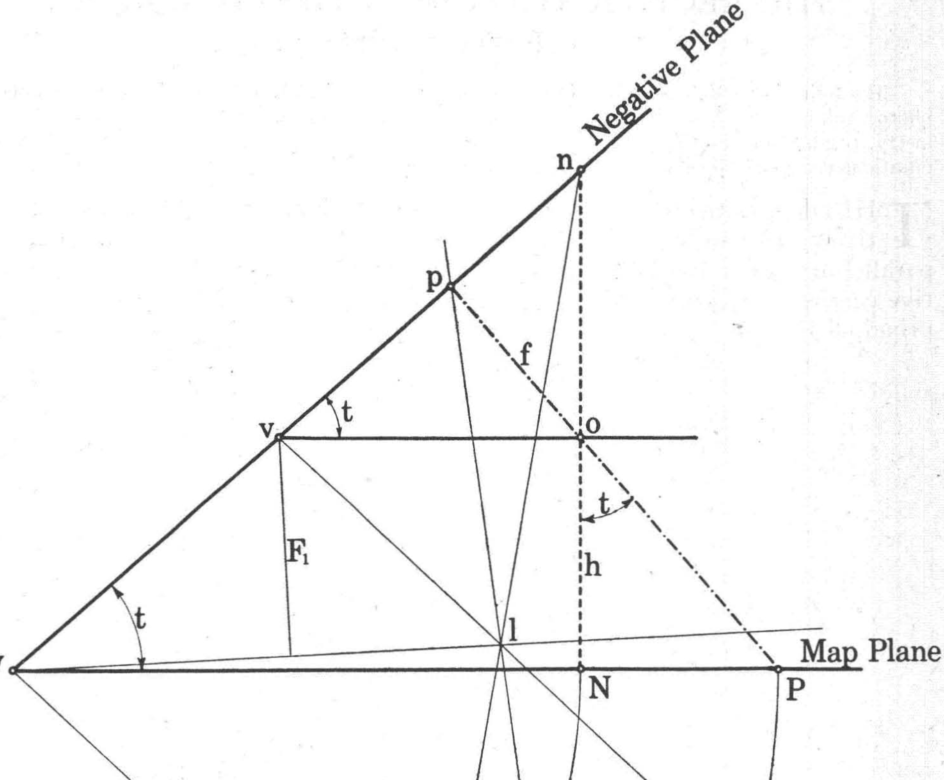

FIG. 2

V represents the intersection of the two picture planes (negative and map planes) in the principal plane of the photograph.

The point *V* is common to the negative and map planes and must therefore also be common to the negative and projection planes. Hence, any position of the projection plane must intersect the negative plane in a line through V perpendicular to the plane of the paper; in other words the projection plane must be swung round this axis through V for any arbitrary new position.

In Figure 2, the map plane is swung to the position VP_1 causing the point P to move to P_1 on an arc whose center is at V.

N is rotated into the position N_1.

The plane Vp will represent the rectifier's negative carrier plane, and VP_1 the easel plane.

It is obvious that the center of perspective (or the lens position of the rectifier) must be at the intersection of the rays passing through pP_1 and nN_1.

The lens position, or center of perspective, therefore is fixed at l.

Figure 1 demonstrates that the negative's vanishing point v lies in a horizontal plane through the lens (o) and parallel to the map plane. This condition follows from the fact that v is the image of the infinitely distant horizon.

Therefore it follows that the projection plane VP_1 in Figure 2 must be parallel to the plane through vl and perpendicular to the plane of the paper for the point v to image at infinity in the projection. Furthermore, focus at infinity demands that v lie in the focal plane of the projection lens l. This condition is fulfilled if the focal length of the projector lens is equal to the perpendicular distance F_1 from the point v to the line lV which represents a trace of the lens plane passing through V to satisfy the Scheimflug condition for sharp imagery.

From similar triangles VPp and vop in Figure 2:

$$\frac{vo}{VP} = \frac{vp}{Vp}$$

hence

$$vo = VP\,\frac{vp}{Vp}\cdot$$

From the similar triangles VP_1p and Vl_1p

$$\frac{vl}{VP_1} = \frac{vp}{VP}$$

but

$$VP_1 = VP$$

therefore

$$vl = VP\,\frac{vp}{Vp} = vo.$$

It follows, therefore, that the locus of the lens (l) as the projection plane is rotated about V is a circle with its center at v and its radius equal to $vo = f \csc t$.

In Figure 1 the negative and map isocenters (n and N) lie on the bisector of the tilt angle NoP (cf. Figure 1). This bisector iI represents a ray through the perspective center o and perpendicular to the bisector of the angle between the negative and ground planes, $PVp = t$.

In the projection diagrammed by Figure 2 this condition must also be fulfilled; that is, the isocenters lie on the ray perpendicular to the bisector of the angle between the negative and projection planes (pVP_1).

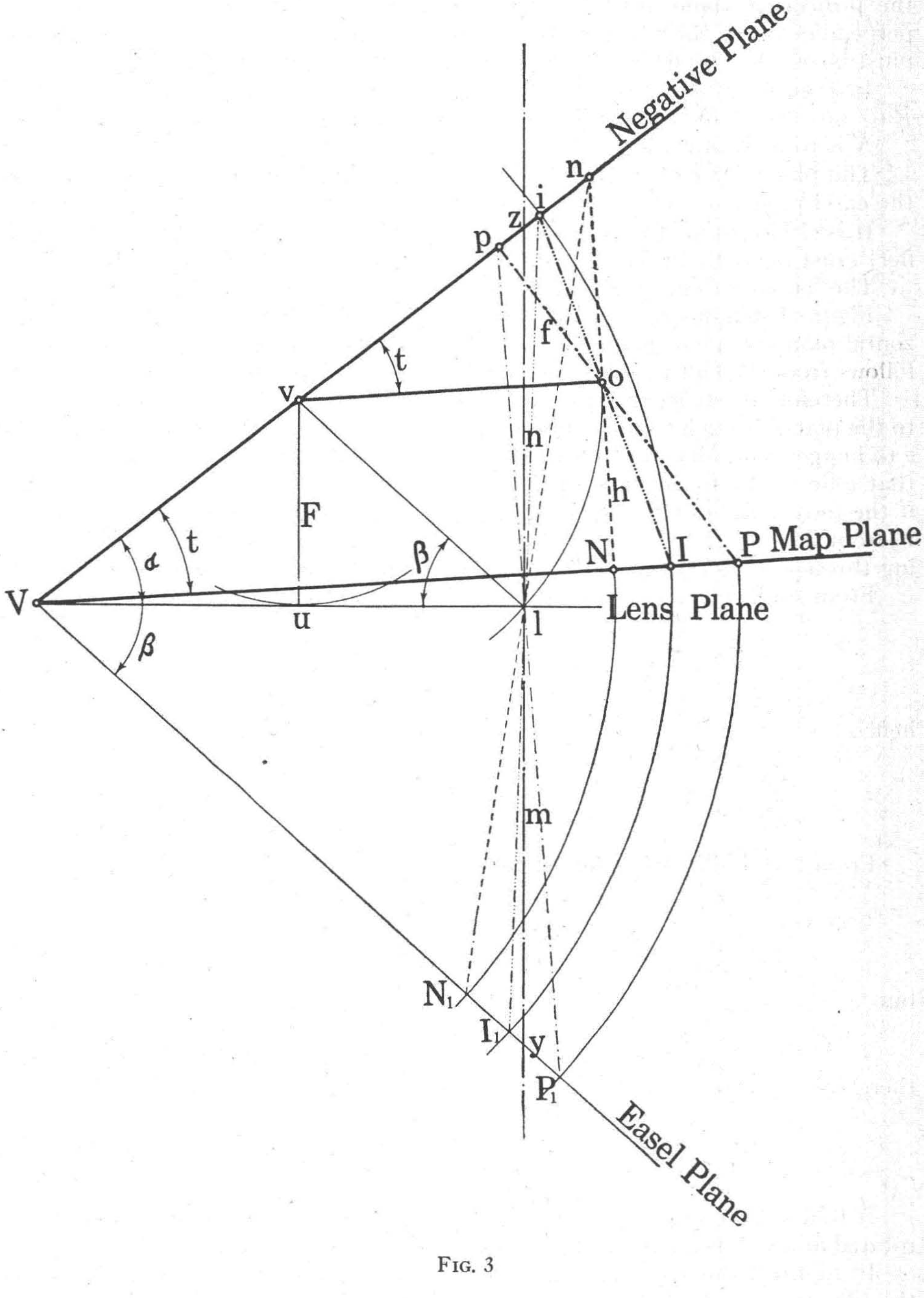

FIG. 3

GEOMETRIC CONSTRUCTION FOR RECTIFIER SETTINGS

Figure 3 shows a graphic solution of the rectification computations; this construction in the principal plane fulfills the optical and geometric conditions previously discussed.

The tilt angle t is established by the intersection of the negative and map planes at V. A horizontal line through the camera lens o intersects the negative plane in the horizon v.

Through V, draw a tangent to a circle with v as center and a radius equal to the focal length F of the rectifier lens. This tangent Vu represents the lens plane. The rectifier lens position is fixed at l, the intersection of the line Vu extended with the arc of a circle having v as its center and vo as the radius.

The easel plane VP_1, is drawn parallel to vl to complete the construction. Its accuracy may be checked by drawing a line through the lens point l perpendicular to the bisector of the angle pVP_1. This line should pass through the isocenters i and I_1.

Formulae to Compute Rectifier Settings

Formulae for computing negative and easel tilts, negative to lens and lens to easel distances and negative displacement are derived from Figure 3.

The negative tilt α is computed from the triangle Vvu

$$\sin \alpha = \frac{F}{Vv}$$

$$Vv = h \csc t$$

therefore

$$\sin \alpha = \frac{F}{h} \sin t. \tag{1}$$

The easel tilt β is computed from the triangle vlw

$$\sin \beta = \frac{F}{vl}$$

$$vl = vo = f \csc t$$

therefore

$$\sin \beta = \frac{F}{f} \sin t. \tag{2}$$

Formula (2) shows that β is independent of magnification.

The negative to lens distance $zl=n$ is derived from the triangle vzl

By the law of sines

$$n = \frac{vl \sin \angle zvl}{\sin \angle vzl}$$

$$vl = F \csc \beta$$

$$\angle zvl = \alpha + \beta$$

$$\angle vzl = 90° - \alpha$$

substituting

$$n = \frac{F \sin (\alpha + \beta)}{\cos \alpha \sin \beta}. \tag{3}$$

The lens to easel distance $ly=m$ is derived from the triangle Vly

$$m = Vl \tan \beta.$$

From triangle Vvu and vul

$$Vl = F(\cot \alpha + \cot \beta)$$

$$= F \frac{\sin (\alpha + \beta)}{\sin \alpha \sin \beta}$$

substituting

$$m = F \frac{\sin (\alpha + \beta)}{\sin \alpha \sin \beta} \tan \beta$$

or

$$m = F \frac{\sin (\alpha + \beta)}{\sin \alpha \cos \beta} \cdot \tag{4}$$

The displacement $d = pz$ of the negative principal point p from the rectifier axis yl is considered negative when downward and positive when upward. In Figure 3 it is shown as negative.

It is evident that,

$$d = vp - vz.$$

From the triangle vop, $vp = f \cot t$

From the triangle vlz by the law of sines,

$$vz = \frac{vl \sin \angle vlz}{\sin \angle vzl}$$

$$vl = f \csc t$$

$$vlz = 90° - \beta$$

$$vzl = 90° - \alpha$$

substituting,

$$d = f \cot t - f \frac{\csc t \cos \beta}{\cos \alpha}$$

$$= \frac{f}{\sin t} \left(\cos t - \frac{\cos \beta}{\cos \alpha} \right) \tag{5}$$

or

$$d = \frac{f}{\tan t} - \frac{F}{\cos \alpha \cdot \tan \beta} \cdot \tag{6}$$

Recapitulation

F = Focal length of the rectifier's lens.
f = Focal length of the aerial camera lens.
h Flying height at the map scale.
t = Tilt of the aerial photograph
α = Tilt of rectifier negative carrier.
β = Tilt of rectifier easel.
n = Distance from negative to rectifiers lens.

m = Distance from easel to rectifier lens.
d = Offset of the negative in the rectifier negative carrier.

$$\sin \alpha = F/h \cdot \sin t \tag{1}$$

$$\sin \beta = F/f \cdot \sin t \tag{2}$$

$$n = F \frac{\sin(\alpha + \beta)}{\cos \alpha \cdot \sin \beta} \tag{3}$$

$$m = F \frac{\sin(\alpha + \beta)}{\sin \alpha \cdot \cos \beta} \tag{4}$$

$$d = \frac{f}{\tan t} - \frac{F}{\cos \alpha \cdot \tan \beta}. \tag{6}$$

If d results in a positive value, the negative must be displaced upward on the negative carrier plane. When d results in a negative value, the negative should be displaced downward on the carrier plane.

The perpendicular on the bisectrix of the angle between the planes of the negative and the easel, drawn through the lens point, will intersect the negative plane at the isocenter of the photograph.

The separation of the nodal points in the rectifier lens does not affect the derivations in this article. However, a correction for the nodal point separation should be applied to the computed settings for the negative and easel distance.

Conclusions

Formulas (1) and (2) show that α and β are not real angles when the ratio F/f and F/h multiplied by $\sin t$ result in values greater than 1. Near these limits a projection will be impractical.

The use of a rectifying lens having a focal length shorter than that of the aerial camera is recommended to avoid excessive tilts of easel and negative carrier. The latitude of the rectifier is increased by using a projection lens having a shorter focal length than that of the aerial camera. Limitations are imposed by the inclinations of the negative and projection planes and by the field angle of the rectifier's lens.

If the negative offset should result in a value which is impossible to recover on the rectifier, it will be found convenient to establish a focal length for the rectifier lens which will permit rectification with a reduced negative offset.

If the offset is 0 for example, the focal length of the rectifier lens must be

$$F = \sqrt{\frac{h^2 \cdot f^2}{h^2 - f^2 \cos^2 t}}$$

and the relationship between the inclinations of negative carrier and the easel will be:

$$\frac{\cos \beta}{\cos \alpha} = \cos t.$$

A lens in the rectifier with a focal length approximately equal to this computed F_1 (when $d=0$) will therefore make a rectification possible because the negative offset will be small.

Graphic Chart for Determining Rectifier Settings

Figures 4, 5, and 6 show the components of a chart which permits graphic determination of rectifier settings.

In practice, the base diagram (Figure 4) is printed on double weight photographic paper. The flight heights diagram (Figure 5) and the displacement diagram (Figure 6) are transparent film positives. These are hinged, individually,

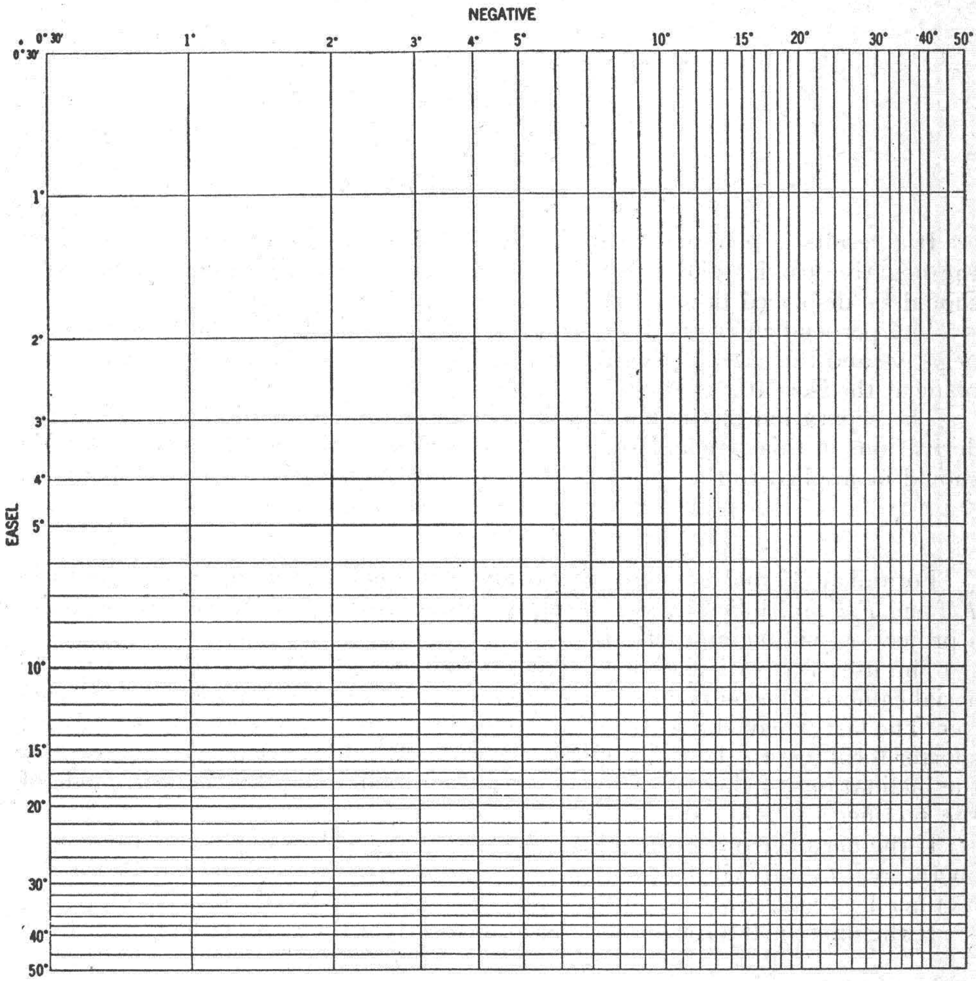

Fig. 4. The base diagram. It will serve for any focal length combination.

to the base diagram, in register. Figure 5 could be overprinted in red on Figure 4 if the chart were prepared by lithography. The displacement or negative offset diagram is used most conveniently as a transparent overlay in either case.

An example will demonstrate the use of the chart. Assume a negative having an air tilt of 9° and a flying height of 320 *m/m* at the scale of the projection. The focal length of the aerial camera is 6″ and the focal length of the rectifier lens is 180 *m/m*. On the right side of the flight height diagram locate the 9° line. Follow this line across to the left hand scale of the base diagram, where easel tilt is read as 10°32′. Again, follow the 9° line across to the left until it intersects the

320 m/m diagonal, thence vertically to the negative carrier scale where the negative tilt is read as 5°02′. The negative offset is read on the overlay at the intersection of the 9° t-line and the 320 m/m h-line. In this example, the offset is plus 1 millimeters which indicates that the negative should be displaced upward from the rectifier axis.

By actual computation, the values are 10°39′, 5°02′, and 1.2 millimeters respectively. The differences between these and the chart values are less than the accuracy with which readings can be recovered in a precision rectifier.

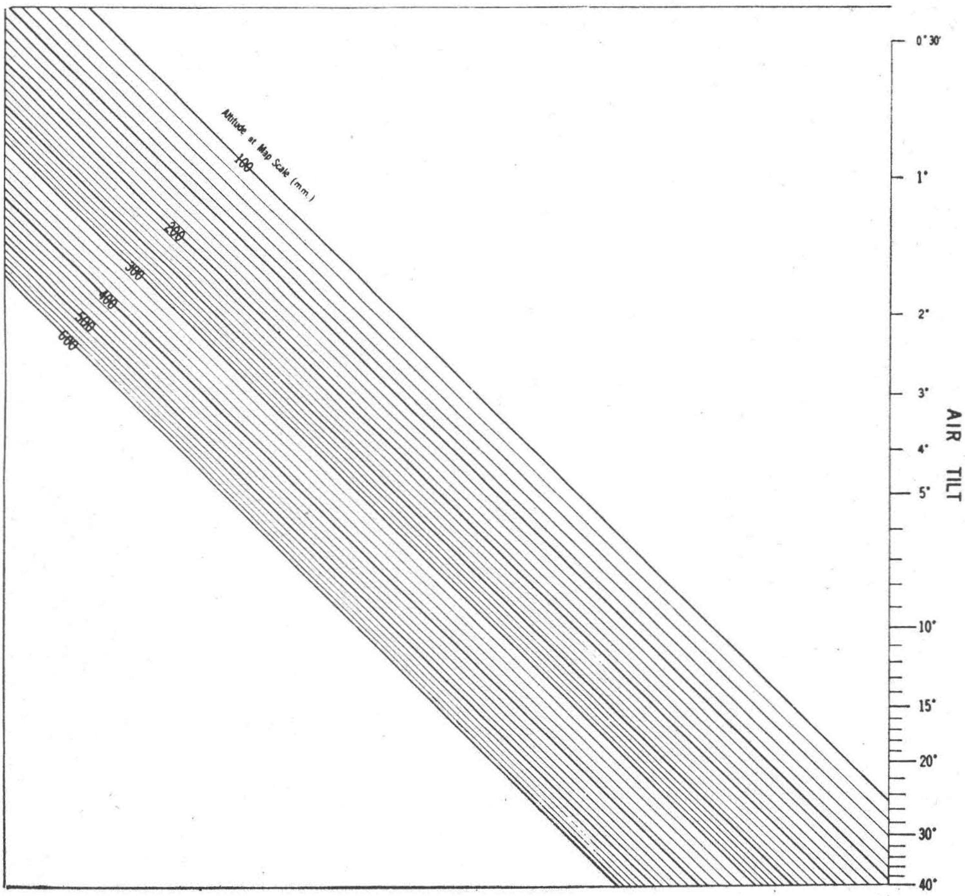

FIG. 5. The flight heights overlay. Focal length of aerial camera = 6″.
Focal length of projector = 180 mm.

The graph applies only to one combination of rectifier and aerial camera lenses. Therefore, it is necessary to prepare a new graph for each new combination.

The scale graduations on the base diagram represent the logarithms of the sines of the tilt angles of the negative carrier and easel. The air tilt scale is identical to the logarithmic sine scales, and its location is given by:

$$\frac{\sin t}{\sin \beta} = \frac{f}{F}.$$

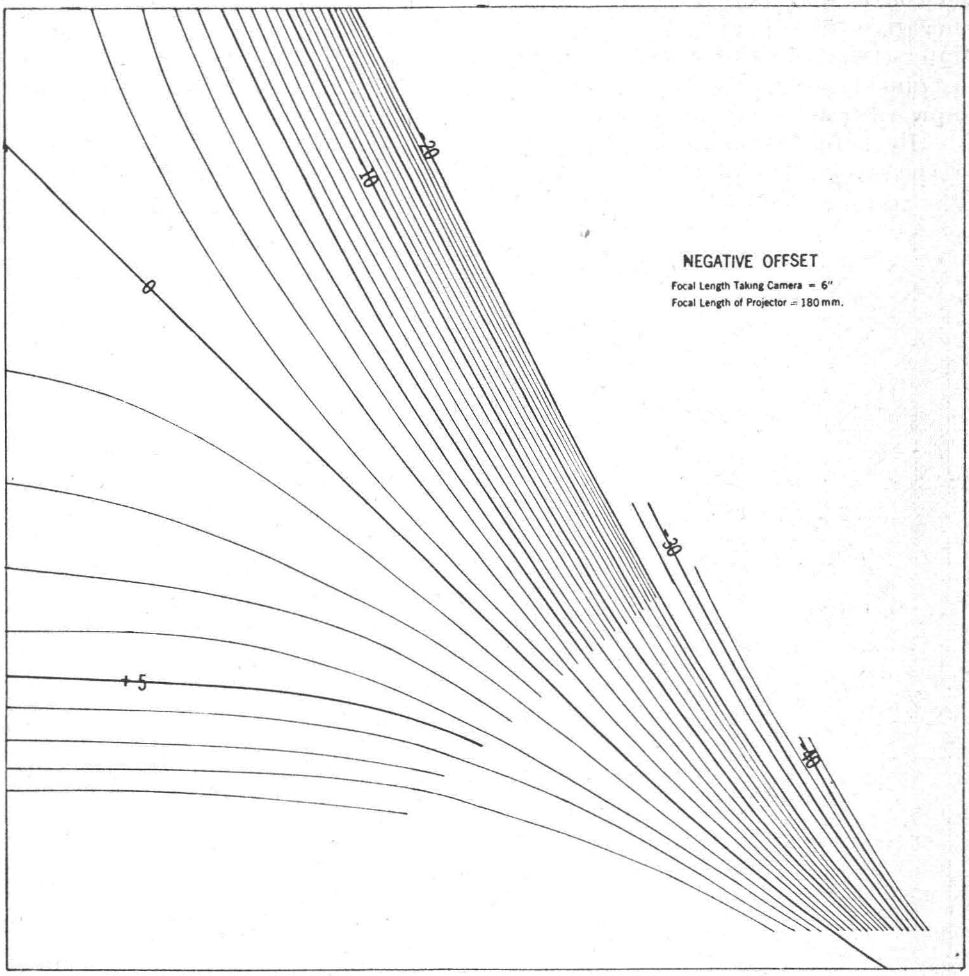

FIG. 6. The displacement or negative offset diagram.

The construction of the flying height graph is illustrated in Figure 7.

It is seen that the flying height diagram construction is not affected by the focal length of the lenses. Its orientation, with reference to the base graph, however does depend on the focal length of the aerial camera lens. By combining the two equations (1) and (2) it is seen that:

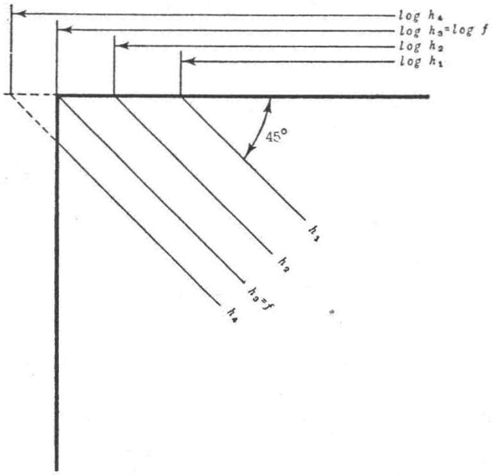

FIG. 7

$$\frac{f}{h} = \frac{\sin \alpha}{\sin \beta}.$$

Therefore, when α is equal to β, then: $h=f$. Consequently, the orientation of the flying height graph over the base diagram must be such that the line representing $h=f$ passes through the upper left and lower right corners of the base graph.

The construction of the offset overlay is done by computing the corresponding value of log sin α to a given tilt of the easel and a given value of the negative's offset. Smooth curves are drawn between points so computed.