From the preceding one might be led to believe that it is all but impossible to make precision maps from aerial photography. On the other hand, acceptable photographs have been made over vast areas of the earth's surface. Also there are instances, when photography is taken over controlled areas, that local disturbances occur in the stereo model. This being true, it must also be assumed that these disturbances occur over the uncontrolled areas. These disturbances are explained by (1) differential shrinkage or distortion of the film; (2) window deformations; (3) possible air turbulence; and (4) in the future high-speed aircraft, the cause may be shock waves.

In weighing the various component's capabilities to produce the degree of precision desired of mapping photography,
the camera apparently has neared its goal. The windows and film bases, however, are under outside influences, which are very difficult to control by man; and, as such, are somewhat difficult to predict at the present time. The making of aerial negatives is the result of the entire operation of the system. Any part being deficient will result in an inferior product. In the correction of the weak points of a system, care must be exercised not to lose sight of any single part of the system, and all effort must not be expended on just one part.

In the future, when more data on the window problem are obtained, and with the advent of improved film bases, the photogrammetrist will receive the type of mapping photography which will make compilation of maps much easier.

# ESTIMATION OF VERTICAL EXAGGERATION IN STEREOSCOPIC VIEWING OF AERIAL PHOTOGRAPHS 

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PUBLISHED articles on this topic give such different results that one is left wondering in what way they are related and what is really correct. Some writers have tried to give a simple working method without explanation, while others have tried to give mathematical proof for statements made. Some made only qualitative evaluation, while others made quantitative studies. Aschenbrenner ${ }^{1}$ reviewed the literature in December, 1952, pointing out conflicts which were largely a result of differences in the points of view of the photo interpreter and of the multiplex operator. Under the circumstances of different points of view and varying purposes, it is not surprising that the results look quite different. It is much more surprising to find conflicting statements about the qualitative effects of different factors. Conflicts are to be found even among three articles published in a single issue of Рhotogrammetric Engineering (Sept., 1953), only a few months after

[^0]Aschenbrenner presumably clarified the situation.

It is the purpose of this paper to examine certain published articles on vertical exaggeration as experienced by an air photo interpreter (or "relief stretching" as Aschenbrenner calls it) to determine the strong and weak points of each, and to evaluate them for accuracy and practicality for use. In conclusion, it is proposed to offer a formula which has general application, but one which is still accurate enough for most purposes and simple enough for easy application. Field testing of different formulas is invited.

## I. A Review of Selected Articles

Kirk H. Stone ${ }^{2}$ chose to call vertical exaggeration, "Appearance Ratio, or ApR." He gives an empirical formula, with no mathematical development, for the ratio of the "apparent height of objects in stereovision to the way they ought to look through the stereoscope." ${ }^{\prime 3}$

[^1]The intent was to give a simple formula which is easy to apply and which gives reasonably accurate results. His formula is quite satisfactory for most purposes if all of his limitations are accepted. He has stated orally that the formula should be used only on $1 / 20,000$ photos of the Production and Marketing Administration. This leaves one desiring a more general formula applicable to other situations, and it arouses questions as to why it needs so many limitations.
"The appearance ratio is computed from four measurements:

## Picture Edge Distance $\times$ Camera Focal Length $\overline{\text { Interpupillary Distance }} \times \overline{\text { Stereoscope Focal-Length }}$

"The picture edge distance is measured along the line of flight between the visible edges of two photos (to be viewed stereoscopically) when they are mosaicked." ${ }^{4}$ The other terms are self-explanatory.

Since much of the air photo coverage of the United States is $1 / 20,000$ scale, flown at 13,750 feet (for the Production and Marketing Administration), a simple method of determining the appearance ratio, or vertical exaggeration, for such photography has real usefulness. It matters little that the formula is empirical so long as it gives the desired results. However, it should be possible to relate an empirical formula which works for a special case to a more general formula which has wider application. On first examination, it is quite disconcerting to find that Stone says that the Appearance Ratio is directly proportional to the camera focal length and inversely proportional to the stereoscope focal length, while other writers who use these factors say that the reverse is true.

Robert F. Thurrell, Jr., ${ }^{5}$ concludes that the Air Base-Height ratio for any given camera focal length and particular size of photograph is the basic relationship in determining vertical exaggeration. He arrived at this conclusion through a number of laboratory experiments and a series of charts.

By definition, Thurrell says, "The amount of exaggeration can be recorded

[^2]as the apparent height of a vertical unit distance divided by the apparent length of an equal horizontal distance." ${ }^{\prime \prime}$

To develop and test his ideas, he used stereoscopic models which were plaster of paris blocks, beveled at angles from 2 degrees to 60 degrees. He photographed the blocks in the laboratory with various heights and base distances, and the baseheight ratio was computed for each stereoscopic pair. The apparent slopes as seen in the stereoscopic models were then estimated by a number of trained photo-interpreters and untrained personnel, and he arrived at an "average" vertical exaggeration. However, the results were consistent enough that, for most people, no large error would be introduced by using an average value.

Thurrell found that the ratio of the tangent of the exaggerated angle to the tangent of the true angle was nearly a constant for each base-height ratio, and the exaggeration factor of height appears as a straight line function of the baseheight ratio. The per cent overlap is uniquely determined by the base-height ratio, the focal length of the camera, and the size of the picture; these things are more easily determined by the photointerpreter. His final chart shows the average exaggeration factor as a linear function of the per cent of overlap, with a different line representing each of several camera focal lengths and picture sizes. ${ }^{7}$ Of the relationship shown in his graph, Thurrell says, "When the size of the print and the focal length of the camera are known, determination of the percentage of overlap allows direct reading of the average exaggeration factor. ${ }^{8}$

By "average exaggeration factor," he means the average for many individuals using the same stereoscope. He recognizes that other factors affect vertical exaggeration, but he treats them as correction factors to the basic base-height ratio. However, the correction factors are the items which are responsible for differences in appearance as seen by different individuals, and they are not negligible factors. Probably the most important of the corrections to be applied is for differences in eye base,
${ }_{7}^{6}$ Ibid., p. 579.
${ }^{7}$ Ibid., p. 586, Fig. 7. Eight lines for different camera focal lengths and picture sizes are given.
${ }^{8}$ Ibid., p. 583 and p. 585.
or interpupillary distance. He found that the average interpupillary distance encountered was $62 \frac{1}{2} \mathrm{~mm}$., and he says that a change of 10 per cent in vertical exaggeration factors is recommended for every 5 mm . variation from average eye base. The exaggeration is greater for a narrow eye base and less for a wide eye base.
One of the principal uses for a vertical exaggeration value is to determine the true angle of slope and a formula for finding the true angle may be written. If $\alpha$ is the true angle and $\beta$ is the apparent angle, then

$$
\tan \alpha=\frac{\tan \beta}{\text { Vertical exaggeration }}
$$

By use of a simple right triangle diagram, it can easily be shown that this formula reduces to an identity.
Having determined the vertical exaggeration and the apparent angle of slope, the true angle can be computed from the above formula. Ordinarily, all of these values are approximations, but for most purposes the approximations are close enough. Table 1 gives the true angle for certain specific vertical exaggerations and apparent angles, given to the nearest half degree. It is apparent from this table that for all cases, except large angles combined with large vertical exaggeration, the error in true angle is relatively small.
Thurrell also notes that a positive correction to the angle is needed if the slope is toward the stereocenter, and a negative correction is needed if the slope is away from the stereocenter, or midpoint between photocenters. He gives a table ${ }^{9}$ showing these corrections. The corrections increase with the size of the angle and with distance from the stereocenter. Unless the slope is quite steep or the location is quite far from the stereocenter, the correction is small. For example, the maximum correction for a 30 degree recorded, or true, slope 1 inch from the midpoint of the two pictures is 1 degree, and it is 5 degrees if 4 inches from the center. Likewise, the corrections for a 70 degree true slope are 4 degrees and 15 degrees for the two respective distances. The maximum correction occurs in the case of the horizontal trace of the slope which is perpendicular to a radial line from the midpoint of the pictures; and there is no correction for a

[^3]Table 1
"True," or Recorded Angle When Estimated Angle and Vertical Exaggeration Are Known
All angles are in degrees to the nearest half degree.

| Esti- <br> mated <br> Angle | $1 \frac{7}{c}$ Vertical Exaggeration |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 2 | $2 \frac{1}{2}$ | 3 | $3 \frac{1}{2}$ | 4 |  |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | $1 \frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 4 | $2 \frac{1}{2}$ | 2 | $1 \frac{1}{2}$ | $1 \frac{1}{2}$ | 1 | 1 |
| 6 | 4 | 3 | $2 \frac{1}{2}$ | 2 | $1 \frac{1}{2}$ | $1 \frac{1}{2}$ |
| 8 | $5 \frac{1}{2}$ | 4 | 3 | $2 \frac{1}{2}$ | $2 \frac{1}{2}$ | 2 |
| 10 | $6 \frac{1}{2}$ | 5 | 4 | $3 \frac{1}{2}$ | 3 | $2 \frac{1}{2}$ |
| 15 | 10 | $7 \frac{1}{2}$ | 6 | 5 | $4 \frac{1}{2}$ | 4 |
| 20 | $13 \frac{1}{2}$ | $10 \frac{1}{2}$ | $8 \frac{1}{2}$ | 7 | 6 | 5 |
| 25 | $17 \frac{1}{2}$ | 13 | $10 \frac{1}{2}$ | 9 | $7 \frac{1}{2}$ | $6 \frac{1}{2}$ |
| 30 | 21 | 16 | 13 | 11 | $9 \frac{1}{2}$ | 8 |
| 35 | 25 | $19 \frac{1}{2}$ | $15 \frac{1}{2}$ | 13 | $11 \frac{1}{2}$ | 10 |
| 40 | 29 | 23 | $18 \frac{1}{2}$ | $15 \frac{1}{2}$ | $13 \frac{1}{2}$ | 12 |
| 45 | $33 \frac{1}{2}$ | $26 \frac{1}{2}$ | 22 | $18 \frac{1}{2}$ | 16 | 14 |
| 50 | $38 \frac{1}{2}$ | 31 | $25 \frac{1}{2}$ | $21 \frac{1}{2}$ | 19 | $16 \frac{1}{2}$ |
| 55 | $43 \frac{1}{2}$ | $35 \frac{1}{2}$ | $29 \frac{1}{2}$ | $25 \frac{1}{2}$ | 22 | $19 \frac{1}{2}$ |
| 60 | 49 | 41 | $34 \frac{1}{2}$ | 30 | $26 \frac{1}{2}$ | $23 \frac{1}{2}$ |
| 70 | $61 \frac{1}{2}$ | 54 | $47 \frac{1}{2}$ | $42 \frac{1}{2}$ | 38 | $34 \frac{1}{2}$ |
| 80 | 75 | $70 \frac{1}{2}$ | 66 | 62 | $58 \frac{1}{2}$ | 55 |
| 90 | 90 | 90 | 90 | 90 | 90 | 90 |

slope for which the horizontal trace is along the radial line from the stereocenter.

In a footnote, ${ }^{10}$ another correction is mentioned for magnification power and viewing distance (eye to photo) of the stereoscope used. Thurrell's study treated these as a constant, but he used a twopower, four and one-half inch focal length pocket folding stereoscope. As variable factors are introduced, such as different stereoscopes or different interpupillary distances, the exaggeration factor is still a straight line function of the base-height ratio, but the slope of the line will vary.

It is unfortunate that Thurrell did not examine the effect of the properties of the stereoscope more thoroughly. It is shown later in this paper that the focal length of the stereoscope and the distance from the lens to the photo have a marked effect on the exaggeration factor.
${ }^{10}$ Ibid., p. 588.

In conclusion Thurrell says, "Vertical exaggeration can be measured in mathematic values which are systematic for all interpreters. The psychological effects expressed by some writers are now outmoded. The stumbling block that remains is the difficulty for the interpreter to estimate with a high degree of accuracy the apparent angles viewed in a stereoscopic model." ${ }^{11}$

Victor C. Miller ${ }^{12}$ made a more qualitative study, and he was interested in vertical exaggeration primarily as a means of determining slope distortion.

Miller says, "It is felt that the general magnitude of vertical exaggeration is largely dependent on the photographic variables and that the stereoscopic variables introduce minor differences in exaggeration." ${ }^{13}$ In addition, he says, "The magnitude of variation in vertical exaggeration and distortion produced by each variable discussed may or may not be significant. Each variable must be subjected to an intensive quantitative test before its relative importance can be known." ${ }^{14}$
E. R. Goodale ${ }^{15}$ gives the impression to a casual reader that his study is made from a strictly mathematical approach and that vertical exaggeration is fully explained by a series of equations and geometric diagrams. Upon more careful reading it is found that he has made some assumptions which seem to the author to be entirely unjustified, and it will be attempted to show in the next section why they are erroneous. He derived a formula which appears to be simple and complete, if the assumptions are granted. He claims that this formula has been tested by various people with satisfactory results. It may be

[^4]that it works satisfactorily for some particular type of photography, but it did not give a reasonable result on any of three types tested by the author, if a correct interpretation of the symbols was made. It gave values of near one-usually a little less than one-for vertical exaggeration. It must be a very unique set of circumstances which will compensate for the errors which appear to be inherent in the formula itself.
The formula for vertical exaggeration as given by Goodale is as follows: ${ }^{16}$
$$
E_{v}=\frac{f_{v}\left(b_{e}+s\right)(b+d)}{f_{a}\left(b_{e}\right)\left(b_{e}+m d\right)}
$$
when
$f_{s}$ is the viewing distance, or stereoscope focal length ${ }^{17}$
$f_{a}$ is the focal length of the air camera lens ${ }^{18}$
$b_{e}$ is the eye base
$s$ is the print separation
$b$ is the photo base
$d$ is the image displacement
$m$ is the magnifying power of the stereoscope.

If $s$ equals $b_{e}$, then the formula reduces to:

$$
E_{v}=\frac{2 f_{s}(b+d)}{f_{a}\left(b_{e}+m d\right)} .
$$

The basic errors in this formula are described below.

1. The first item is more misleading than erroneous. In the explanation of the diagrams from which the formula was developed, Goodale defines $f_{s}$ as the stereoscopic viewing distance, which is correct. However, in the explanation of the formula, as shown above, he says that $f_{s}$ is the viewing distance, or stereoscope focal length. This implies that the two are the same. This is impossible because this is the one position for the object which gives no image whatever for a convex lens (or for a plano-convex lens as the Fairchild folding stereoscope is). In order to have magnifica-

[^5]tion and have the image on the same side of the lens as the object, the object (photograph) must be nearer the lens than the focal length. Therefore, since Goodale's formula is based upon a pseudogeometric development from his diagrams, it must be understood that $f_{s}$ represents the viewing distance, or object distance, and not the focal length of the stereoscope.
2. Goodale makes an assumption that the eye and mind somehow bring the images through the two eye-pieces into stereovision on some plane which he calls the "Plane of Stereoscopic Fusion of Points." He is unable to locate this plane, but he feels that it lies somewhere not far below the plane of the photograph. Actually, he assumes that the plane for the low, or datum, point lies the same distance below the photograph as the lens is above it. The most casual viewing will indicate that this is not true. If one will look at a pair of pictures stereoscopically, and at the same time place his hand at about the level where the image seems to lie, he will easily discover that it is much farther below the picture than the distance that the lens is above the photograph. (Using a folding pocket stereoscope, the author estimated that the image was about 18 inches below the picture by this method long before he found any proof of it.) Another device which helps to show that the image is much farther below the photo plane than the stereoscope is above it, is to fold the legs of the stereoscope and move the stereoscope up and down within the range which gives clear stereovision. It will be seen that the image moves in the direction opposite to that of the stereoscope, but at a much faster rate.
3. From his diagrams, Goodale geometrically develops the equation,
$$
n_{s}=K \frac{f_{s} \cdot s}{b_{e}}
$$
where $K$ is the proportionality factor and $n_{s}$ is the distance from the photo plane to the image plane (or plane of fusion, as he calls it) for the datum point. This appears to be correct from the drawing, if we understand that the line labeled $f_{s}$ is actually the object distance instead of the focal length of the stereoscope. (In Figure 1, the image distance is called $d_{i}$ and the object distance is called $d_{o}$.)

Goodale says, "The value of $K$ may well
be determined experimentally, by finding a direct way of measuring $n_{s}$. However in using (the above) equation, it is assumed that $K$ has a value of one (1), inasmuch as this agrees with the geometry of figure 4-b." ${ }^{19}$

If $K$ equalled 1 , then $n_{s}$ would be approximately equal to the object distance, but actually less under the best viewing condition which has $s$ slightly less than $b_{e}$. This is contrary to principles from elementary physics regarding the location of images.
4. Goodale's location of the "plane of fusion," or the location of the image in stereovision, is based largely upon his assumption that the eye-base and photo separation are equal. He then hypothesized that the eyes and mind somehow combine in a "plane of fusion" the light of parallel rays coming from corresponding points of the two photos. Goodale, and some others, claim that it is even possible to see stereoscopically by divergent rays. The author finds that this is impossible for himself. However, there is no question but that human vision does have power of adjusting to a wide range of conditions. It is possible to see stereoscopically under a variety of adjustments, just as it is possible to see through a pair of glasses even though they are very poorly suited to the eyes of the wearer. The author maintains that there is one adjustment of the stereoscope and of the pictures which gives the clearest and best stereovision with the least eye-strain, and that using such an adjustment is the proper way to do stereoscopic viewing. It is also maintained that, when stereoscopic viewing is done properly, the photo separation is always less than the eye base. More strictly, the separation is always less than the distance between the centers of the lenses. However, for best viewing, the distance between the lenses is equal to the eye base. If not, the view is distorted, and one uses only a small part of the lens. If the image plane is drawn far below the photo plane in a schematic diagram as it should be, then lines from the eyes, and the lens centers, through corresponding points on the photos will converge with the proper order of magnitude in $b_{e}-s$. This will be shown numerically later after proposed changes in Goodale's formula are given.

[^6]
## II. Proposed Changes in Goodale's <br> Formula to Make It Compatible with Physical Principles

First, the author believes that the phenomenon of stereovision can only be explained by actual intersection of rays of light as seen by the two eyes and extended through the corresponding points on the two photographs. Goodale denies this, but he has no logical theory to substitute for it.

Second, the author believes that the print separation is always less than the distance between the centers of the stereoscope lenses if distinct stereovision is to be seen. Also, the best image is seen when the lens centers are separated by a distance equal to the eye-base. Therefore, the photo separation for best stereovision is less than the eye base, and the light rays do converge in a definite and measurable pattern. These statements are made on the basis of repeated personal observations and measurements. Goodale has the separation equal to the eye base, and hence he has all rays of light to the datum plane as non-converging. Therefore, in his explanation he was forced to use a "plane of fusion" which he could not locate. It is true that light rays to points of different elevation do not intersect in the same horizontal plane. However, for objects of moderate height, we get near enough to coincidence of images that both top and bottom of the object appear to coincide at the same time, or at least there is near coincidence. Light rays for high points intersect somewhat above the datum image plane, and the rays for points below the datum plane will intersect below the image plane. Intersections either above or below the datum plane are not in sharpest focus although they may be clearly distinguishable, If viewing an area of rough terrain, it may be necessary to use a different print separation for the ridges than for the valleys to bring each into sharp focus. If this is done, the wider separation is used for the ridges. In the case of a very tall object, it may not be possible to get the top of the object to coincide while the datum plane is in focus. If the top part is brought into clear stereovision by moving the prints farther apart, the bottom part of the building or object may be out of focus and out of coincidence. This is known as "split" stereovision, and results only if
the image displacement is sufficiently large that not even illusionary coincidence can be attained.

Third, in every place that $f_{s}$ occurs in Goodale's drawings and equations, substitute " $d_{o}$," or object distance, to make sure that this distance is not mistakenly taken as the focal length of the stereoscope. When the object (photo) is at the principal focus (that is, when $d_{o}=f_{s}$ ) the rays of light are parallel as they leave the lens, and no image is formed. By actual measurement on the stereoscope in use, the focal length was $4 \frac{1}{2}$ inches and the object distance was $3 \frac{5}{8}$ inches. Therefore, the photo is nearer the lens than the principal focus, and the image is virtual, on the same side of the lens as the photo, erect, and enlarged.
Fourth, the distance from the photo to the image plane is a definite length dependent only upon the stereoscope in use and the distance it is held from the photo plane. From elementary physics,

$$
\frac{1}{d_{0}}+\frac{1}{d_{i}}=\frac{1}{F},
$$

where $d_{o}$ is the object distance, $d_{i}$ is the image distance, and $F$ is the focal length of the lens. If $d_{i}$ is found to be negative, as is the case for lens stereoscopes, the image is virtual.
For the particular stereoscope used, a Fairchild, Model C2, the object distance when the stereoscope is sitting on its legs is $3 \frac{5}{8}$ inches and the focal length is $4 \frac{1}{2}$ inches. Substituting in the formula,

$$
1 / 3.625+1 / d_{i}=1 / 4.5
$$

whence $d_{i}$ is 18.64 inches.
The image plane is therefore 18.64 inches below the lens, or $18.64-3.625$, or approximately 15 inches below the photo plane. This is the distance which Goodale calls $n_{8}$. In the formula which Goodale has for $n_{s}$, the $K$ factor of proportionality is therefore approximately 4.5 instead of 1 as he assumed. However, since we now have a direct method of finding $n_{s}$, it is no longer necessary to use $K$.
Solving the above formula for the image distance, $1 / d_{i}=1 / F-1 / d_{0}$, and

$$
d_{i}=\frac{F \cdot d_{0}}{d_{0}-F}
$$

For the ordinary stereoscope, $d_{i}$ is negative, which merely indicates direction. In
order to avoid confusion of negative signs, let us use only the absolute value of $d_{i}$, or to make it positive, write

$$
d_{i}=\frac{F \cdot d_{0}}{F-d_{0}} .
$$

$F$ is the focal length of the stereoscope. In order to make our notation more consistent, let us use $f_{s}$ to represent the focal length of the stereoscope and write

$$
d_{i}=\frac{f_{s} \cdot d_{0}}{f_{s}-d_{0}} .
$$



Fig. 1. Diagram of a vertical section of a stereoscopic model showing relationship of object and image rays when the images of the low (datum) point are in coincidence, and assuming no separation of the photos. ${ }^{21}$
$b$-Photobase, or scale distance of air base.
$b_{e}$-Eyebase, or interpupillary distance.
d -Sum of image displacements of high point $u$ referred to photographic datum.
$d_{o}$-Object distance, or stereoscopic viewing distance.
$h$-True height of high point $u$ at photographic datum scale.
$h_{o}{ }^{\prime}$-Stereoscopic height of high point, $u$, with no separation.
$m d$-Image displacements, $d$, times magnifying power, $m$, of stereoscope.
$u$-Point of intersection of object rays of high point.
$u^{\prime}$-Point of intersection of image rays of high point.
$f_{a}$-Focal length of air camera lens.

From Figure 1,

$$
h=\frac{f_{a} \cdot d}{b+d}
$$

and

$$
h_{0}^{\prime}=\frac{d_{0} \cdot m d}{b_{e}+m d},
$$

following the method of Goodale ${ }^{20}$ except that $d_{0}$ is used instead of $f_{s}$.

From Figure 2,

$$
\frac{h_{0}^{\prime}}{h_{s}^{\prime}}=\frac{d_{0}}{d_{0}+n_{s}}
$$

and therefore

$$
h_{s}^{\prime}=\frac{h_{0}{ }^{\prime}\left(d_{0}+n_{s}\right)}{d_{0}} .
$$

Since $d_{0}+n_{s}$ equals $d_{i}$,

$$
h_{s}^{\prime}=\frac{h_{0}^{\prime} d_{i}}{d_{0}} .
$$

Substituting for $h_{0}{ }^{\prime}$ and $d_{i}$, this becomes:

$$
h_{s}^{\prime}=\frac{m d f_{s} d_{0}}{\left(f_{s}-d_{0}\right)\left(b_{e}+m d\right)} \cdot
$$

By definition, vertical exaggeration is the ratio of the apparent height of a vertical unit distance to the apparent length of an equal horizontal distance; or as Stone puts it, it is the ratio of the apparent height as seen through the stereoscope to the way it ought to look through the stereoscope. Thus, vertical exaggeration,

$$
E_{v}=\frac{h_{s}^{\prime}}{m h}
$$

Substituting the values given for $h_{s}{ }^{\prime}$ and $h$, and simplifying,

$$
E_{v}=\frac{f_{s} d_{0}(b+d)}{f_{a}\left(f_{s}-d_{0}\right)\left(b_{e}+m d\right)}
$$

where
$E_{v}$ is vertical exaggeration.
$d_{0}$ is the distance from the lens to the photo, or the object distance.
$f_{s}$ is the focal length of the stereoscope.
$b$ is the photo base, or "picture-edge distance."
${ }^{20}$ Ibid., pp. 613 and 615.
${ }^{21}$ Diagram based on Figure 3 from E. R. Goodale, "An Equation for Approximating the Vertical Exaggeration of a Stereoscopic View."
$f_{a}$ is the focal length of the camera.
$b_{e}$ is the eye-base, or interpupillary distance.
$m$ is the magnifying power of the stereoscope.
$d$ is the image displacement. This is the sum of the two displacements on the two photos, measured along a line parallel to the flight line.
The above formula is to be used only for vertical photography and for a lens stereoscope. Presumably, a very similar formula could be developed for a mirror


Fig. 2. Diagram of vertical section of stereoscopic model showing pattern of light rays when photos have separation $s$.
$b_{e}, d_{o}$, and $h_{o}{ }^{\prime}$ are used in the same way as in Figure 1.
$d_{i}$-Image distance from lenses.
$h_{s}^{\prime}$-Apparent height of object of height $h$.
$m d_{l}$-Image displacement times magnification as seen by left eye.
$m d_{r}$-Image displacement times magnification as seen by right eye. The sum of $m d_{l}$ and $m d_{r}$ is equal to $m d$ as in Figure 1 .
$n_{s}$-Distance of image below photo plane.
$s$-Separation of photos.
stereoscope, differing primarily in the formula for the image distance, but the author has done no work on that.

Some of the terms in the formula above are constant for a particular type of stereoscope, while others are constants for a particular type of photography. By substituting the appropriate constants this cumbersome looking formula can be greatly simplified for a specific use. To illustrate, let us first make simplifications for a particular stereoscope. There is also assumed a magnification power of 2 , a focal length of 4.50 inches $\left(.375^{\prime}\right)$, and one which sits 3.625 inches ( $.302^{\prime}$ ) above the photo.

The formula then reduces to:

$$
E_{v}=\frac{18.64(b+d)}{f_{a}\left(b_{e}+2 d\right)}
$$

if the units are in inches, or

$$
E_{v}=\frac{1.551(b+d)}{f_{a}\left(b_{e}+2 d\right)}
$$

if the units are in thousandths of a foot.
For a particular camera focal length, the formula may be further reduced. If we have a camera focal length of $8.25^{\prime \prime}$ and a stereoscope as assumed above,

$$
E_{v}=\frac{2.26(b+d)}{b_{e}+d},
$$

regardless of the units used.
The formula now looks quite simple except for the term, $d$. It is difficult to measure $d$ with much accuracy. In Figure $1, d$ is the sum of the displacements on the two photos, and $m d$ is the magnified sum of the two displacements. It was from the geometry of Figure 1 that $d$ and $m d$ entered the formula. For a tall object, it is possible to find the difference between the print separation for the top and the bottom of the object. This difference is $d$ if measured with the naked eye and $m d$ if measured under the stereoscope. For objects of moderate height, the difference is too small to distinguish clearly, unless the scale of the picture is quite large.
Let us investigate the magnitude of $d$ for a $1 / 20,000$ photo taken with a camera of $8.25^{\prime \prime}$ focal length. First, it should be noted that $d$ is variable according to the height of the object, the distance from the center of the photo, the slope of the land upon which it stands, and the direction of that slope with regard to the flight line. Thus, strictly speaking, there is a different $d$ for every point in the picture which is
above or below the datum plane. However, it can be shown that for most cases the difference is negligible. The flight elevation for a $1 / 20,000$ photo taken with a camera of focal length of $8.25^{\prime \prime}$ is $13,750^{\prime}$. Let us assume that the land is level and that there is an object $100^{\prime}$ located $5,000^{\prime}$ from the point below the camera, measured along the line of flight.

By a simple right triangle diagram it can be shown that for this case the line of sight from the camera to the top of the $100^{\prime}$ object intersects the ground at a distance of $36.36^{\prime}$ from the base. This will show on the photo as a distance of $.0018^{\prime}$, or $.0216^{\prime \prime}$. This is the displacement on only one photo. However, this is almost the maximum possible displacement on a $7^{\prime \prime} \times 9^{\prime \prime}$ photo of $1 / 20,000$ scale as such a point is near to the edge of the photo. In such a case, the displacement on the other photo is very small. Therefore, the maximum displacement, as a sum for the two photos, is very close to $.002^{\prime}$ under conditions of the example, regardless of the location on the photo, and the average displacement is much less. Therefore, for most practical purposes, the value of $d$ may be ignored, at least for heights which are less than about $2 \%$ of the flight altitude, and probably for any height which can be seen clearly without "split" stereovision.

If $d$ is regarded as negligible, although always keeping in mind that a larger value of $d$ makes a slightly smaller vertical exaggeration, the formula developed in this paper strongly resembles Stone's formula, if the ratio of the focal lengths were expressed as a constant. Using the notation which is used in the formula developed in this paper, Stone says:

$$
E_{v}=\frac{f_{a} b}{f_{s} b_{e}} .
$$

It is unfortunate that Stone implies in his empirical formula that vertical exaggeration is directly proportional to the camera focal length and inversely proportional to the stereoscope focal length. The other three articles which have been reviewed and the present paper show that vertical exaggeration is directly proportional to the stereoscope focal length and inversely proportional to the camera focal length. It would cause much less confusion if vertical exaggeration were expressed as a constant times the ratio of $b / b_{e}$, the constant depending upon the type of
stereoscope and focal length of the camera. For a camera focal length of $8.25^{\prime \prime}$ and a stereoscope focal length of $4.5^{\prime \prime}$, the constant which would result from Stone's formula is 1.83 . The corresponding constant by the formula of this paper is 2.26 .

## III. Examples of Applications to Various Types of Photography

A few examples are given to show the relative values found for vertical exaggeration by the four methods discussed in this paper. Since Miller's study is not quantitative, no results are obtainable which correspond to his work.

Constants used for all examples:

$$
\begin{aligned}
& b_{e}: .206^{\prime} \text { or } 2.47^{\prime \prime} \\
& d_{0}: .302^{\prime \prime} \text { or } 3.62 .625^{\prime \prime} \\
& f_{8}: .375^{\prime} \text { or } 4.5^{\prime \prime}
\end{aligned}
$$

${ }_{d}$ is assumed to be negligible and this is justifiable for attaining a general exaggeration factor. However, if the object in which there is a special interest is quite tall, then $d$ should be measured, as it certainly is not zero.
Example 1. $7^{\prime \prime} \times 9^{\prime \prime}$ photo of $1 / 20,000$ scale; $b$ of $2.46^{\prime \prime}$ or $.205^{\prime}$; $s$ of $2.16^{\prime \prime}$ or . $180^{\prime}$; overlap of $65 \% ; f_{a}$ of $8.25^{\prime \prime}$ or $.6875^{\prime}$.
Vertical exaggeration according to:

| Thurrell: | 2.0 |
| :--- | :--- |
| Goodale: | 0.82 |
| Stone: | 1.83 |
| Treece: | 2.26 |

Example 2. $9^{\prime \prime} \times 9^{\prime \prime}$ photo of $1 / 20,000$ scale; $b$ of $.284^{\prime} ; s$ of $.188^{\prime}$; overlap of $61 \%$; $f_{a}$ of .6875'.

Vertical exaggeration according to:
Thurrell: 2.8
Goodale: 1.2
Stone: 2.5
Treece: 3.1
Example 3. $9^{\prime \prime} \times 9^{\prime \prime}$ photo of $1 / 9,600$ scale; $b$ of $.340^{\prime} ; s$ of $.186^{\prime}$; overlap of $56 \%$; $f_{a}$ of 1 foot.
Vertical exaggeration according to:
Thurrell: 2.2

* Goodale: 1.35

Stone: Formula does not apply Treece: $\quad 2.63$

In making the above computations for vertical exaggeration by Goodale's formula, $f_{s}$, is taken as the object distance $\left(d_{o}\right)$ as it is in the diagrams from which he derives the formula. However, if $f_{s}$ in his formula is taken as the focal length of the
stereoscope as he writes part of the time, then the vertical exaggeration is increased by a factor of 1.25 . For Example 1, this would make the $E_{v}$ equal to 1.0 ; it would be 1.5 for Example 2; and 1.7 for Example 3. Only in Example 3 does Goodale's value get much above 1. If the correct interpretation was given to his symbolism, his formula is unsatisfactory both from the standpoint of application and of theory.
Thurrell's method was difficult to derive experimentally; but once he has reduced the findings to a graphical representation, it is very easy to apply and the results seem reasonable. For the two examples using $8.25^{\prime \prime}$ focal length, and to which Stone's method applies, Thurrell's value lay between those of Stone and Treece. The only difficulty with his method is the application of the correction factors which he mentions. In the values given above, no correction factors were applied.

Stone's formula is very limited in the type of photography for which it is recommended, and even for that type the vertical exaggeration values appear to be on the low side, but not enough to make a great difference in the value obtained for the angle of slope in most cases. It is highly recommended though that he give his formula as a constant times the ratio of $b / b_{e}$ instead of giving an incorrect impression of the effect of the stereoscope and camera focal lengths. If the findings of this paper are correct, he also needs to revise that constant upward to give a more true value.

The values found by the formula derived in this paper were the highest of any found for all three examples. This may or may not indicate something defective with it. The author thinks not. His own limited experience with estimating angles with a
stereoscope, and then measuring them in the field, is that the estimation usually is too high. A larger vertical exaggeration factor would tend to correct the error. The fact still remains that the most difficult part of measuring slope with a stereoscope is the estimation of the angle of slope as it appears under the stereoscope.
Others are invited to test the results obtained by using the various methods described. The photo-interpreter is urged, however, to estimate the angles as they actually appear without making any kind of a mental correction to make them what he thinks that they ought to be. The results should then be field-tested. It is hoped that some one will report on the comparison of the methods as tested in the field.

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## DISCUSSION OF PAPER BY WALTER A. TREECE

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MR. TREECE's paper is interesting and most welcome. There is little doubt that it will promote renewed discussion of a subject which requires and surely deserves more profound and thoughtful study. In my opinion, we have had enough theorizing. What is needed now are good, sound experimentation and proofs.

In this respect I find Mr. Treece's paper lacking. There are two drawings given to illustrate geometrical relationships, one of which is admittedly copied from another paper, the other smacks of the pen of Wheatstone, who offered his concept of stereoscopy to the world more than 100 years ago.


[^0]:    ${ }^{1}$ Aschenbrenner, Claus M., "A Review of Facts and Terms Concerning the Stereoscopic Effect," Photogrammetric Engineering, Dec., 1952, pp. 818-823.

[^1]:    ${ }^{2}$ Stone, Kirk H., "Geographical Air-PhotoInterpretation," РнотоGrammetric Engineering, Dec., 1951, pp. 754-759.
    ${ }^{3}$ Ibid., p. 757.

[^2]:    ${ }^{4}$ Ibid.
    ${ }^{5}$ Thurrell, Robert F., Jr., "Vertical Exaggeration in Stereoscopic Models," Рhotogrammetric Engineering, Vol. XIX, Sept., 1953, pp. 579-588.

[^3]:    ${ }^{9}$ Ibid., p. 587.

[^4]:    ${ }^{11}$ Ibid.
    ${ }^{12}$ Miller, Victor C., "Some Factors Causing Vertical Exaggeration and Slope Distortion on Aerial Photographs," Рнотogrammetric Engineering, Vol. XIX, Sept., 1953, pp. 592-607.
    This article is similar to his earlier one in 1950: "Rapid Dip Estimation in Photo-Geological Reconnaissance," Bulletin American Association of Petroleum Geologists, Vol. 34, No. 8 (August, 1950).
    ${ }^{13}$ Ibid., p. 592.
    ${ }^{14}$ Ibid., p. 607.
    ${ }^{15}$ Goodale, E. R., "An Equation for Approximating the Vertical Exaggeration Ratio of a Stereoscopic View," Photogrammetric Engineering, Vol. XIX, Sept., 1953, pp. 607-616.

[^5]:    ${ }^{16}$ Ibid., p. 610.
    ${ }^{17}$ These two things can not be equal. See text above for explanation.
    ${ }^{18} f_{a}$ also corresponds to the height of the airplane, reduced to the scale of the picture. It is sometimes more helpful in visualizing the pattern of light rays to think of $f_{a}$ as the scaleheight rather than focal length of the camera.

[^6]:    ${ }^{19}$ Goodale, op. cit., p. 615.

