Measurement of Vertical Heights from Single Oblique Aerial Photographs

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ABSTRACT: *Nomographic methods for indirect measurement of vertical heights from images appearing on single-oblique aerial photographs are presented in this paper. The application of the methods is quite elementary and may be learned by non-technical personnel in a matter of minutes. A measurement of a height may then be performed in two or three minutes with an accuracy of a very few per cent, depending primarily on the accuracy of photographic parameters and the skill of the operator.*

1. INTRODUCTION

IN A previous paper presented in this journal* the author described a method for performing indirect measurements of ground objects from single-oblique aerial photographs utilizing a set of nomographic charts. The measurements presented in that paper were limited to objects lying in a level plane—linear dimensions, areas, and angles. The same approach was subsequently applied to measurement of vertical heights, and it was soon found that the method was even more suitable for this than for level measurement. Image parameters are reduced to two, and the number of nomographs required is reduced to one. The method is nearly, though not quite, as simple as the corresponding technique used with vertical photographs in which the object height,

$$h = \frac{l}{d} \times \text{altitude}, \qquad (1)$$

where l is the image length of the height and d is the distance from the top of the image to the principal point of the photograph.

Early in the development of the nomographic method for measuring vertical heights, it became apparent that due to the character of different height images on a photograph, two solutions would be necessary—one for heights projecting short images on the photograph, and another for long images. Therefore, two different methods, equally simple, and each utilizing a separate nomograph are presented in this paper. The difference between "long" and "short" images is determined by the practicality of accurate measurement of the image and accurate manipulation of the image parameters on the nomograph.

The nomograph for long images may also be used to measure a component of the ground distance of a point from the nadir point of the aircraft, at the time of exposure of the photograph.

Either height-measuring method may be taught in ten or fifteen minutes to non-technical personnel having little mathematical background. The principal skill required is ability to use a pair of dividers and to multiply two or three numbers together. An operator proficient in the method should be able to measure a ground height from a photo image in about one or two minutes. Accuracy should be within a very few per cent, depending principally on accuracy of measurement of the image length.

The foregoing is based, of course, on the premise that the operator is provided a photograph with accurately known parameters of tilt, altitude, and focal-length, on which are marked accurate photographic axes. Methods of determining these parameters are found, among other places, in the MANUAL OF PHOTOGRAMMETRY and in the aforementioned paper by the author, the latter containing several short-cuts, utilizing nomographs.

* Gay, S. P., Jr., "Nomographic Solution to Oblique Photo Mensuration," PHOTOGRAMMETRIC ENGINEERING, Vol. XXII, no. 4, pp. 674–692.

One limitation of the nomographic methods presented in this paper is that the height to be measured must be a vertical height, such as a flagpole, smoke-stack, or corner of a building. It will not measure relative elevations of different objects as will parallax methods utilizing stereoscopic pairs of photographs.

II. Measuring Heights from Short Images

The *height-factor chart* (Figure 1) is the nomograph used in measuring vertical heights on an oblique photograph when the image length is small, On this chart a number, or *height-factor*, is determined using image and photographic parameters. The height-factor multiplied by the image length and the quotient of the altitude and focal-length gives the correct height of the object being measured:

$$\text{Height of object} = \frac{\text{altitude}}{\text{focal length}} \times \text{image length} \times \text{height factor} \qquad (2)$$

To determine the proper height-factor for a photographic image the photo interpreter simply measures the perpendicular distance from the center of the image to the x-axis of the photograph, i.e., the *vertical displacement* of the image. This distance is then located on the scale for the proper focal-length at the left edge of the height-factor chart, and a line is extended horizontally until it intersects the curve for the correct depression-angle of the photograph (images above x-axis) or tilt-angle (images below x-axis). At this intersection, the height-factor is read between vertical lines on the scales across the top and bottom of the chart.

When the height-factor chart is reproduced at the scale corresponding to the correct focal-length for a photograph, the operator may simply take the vertical displacement of the image with dividers and apply it to the chart in one step. For the other focal-lengths shown, a pair of proportional dividers may be used in scaling off the distance. This method may also be used for photographs of any focal length. The ratio is an inverse one. Thus, the chart coordinate for an image point on a photograph of 7″ focal-length would be 12/7 of its value for an image point of the same vertical displacement on a photograph of 12″ focal-length.

The height-factor chart, then, is universal in its use. It is not restricted to any particular depression-angle or focal-length. It requires no knowledge of mathematics except as required by formula (2). The manipulation of the chart is so elementary that few advantages would accrue from specialization of the chart, such as adaption for one particular focal-length and depression-angle. The formula used with the chart, formula (2), may be recognized as the same as that used for computing horizontal distances on a map or on a vertical photograph—multiplied by the height-factor. This same formula is also used for computing horizontal distances on oblique photography with the authors nomographic method, except that in this case, an "oblique" or level-line correction-factor is used instead of the height-factor.

The height-factor chart in its present form is restricted, however, to photographic images lying above, or beyond, the nadir point. On oblique photographs of low tilt-angle, there is quite a sizeable photographic area lying between the nadir parallel and the lower edge of the photograph, and it may occasionally be desirable to measure heights in this region. Large heights may be measured on the nomographic chart described in the following section, but it was not felt that the need for measuring small images here justified construction of additional curves on the height-factor chart. Furthermore, near the nadir-point where the camera viewpoint is almost vertical, the image-length is necessarily very small and the height-factor large, and the accuracy of the nomographic method rapidly decreases. Parallax methods using stereo pairs of photographs are much more applicable to this situation. Such methods for oblique photography are presently under development by the author.

There is another important qualification in the application of the nomographic method. It has been implied that a direct measurement of the photographic image is made. This is only true for those small images far from the nadir-point of the aircraft and near the principal line of the photograph which are essentially parallel to the principal line. As images of vertical lines on an aerial photograph always point to the nadir, then many images on the sides of the photograph will be quite non-parallel with the principal line. For these

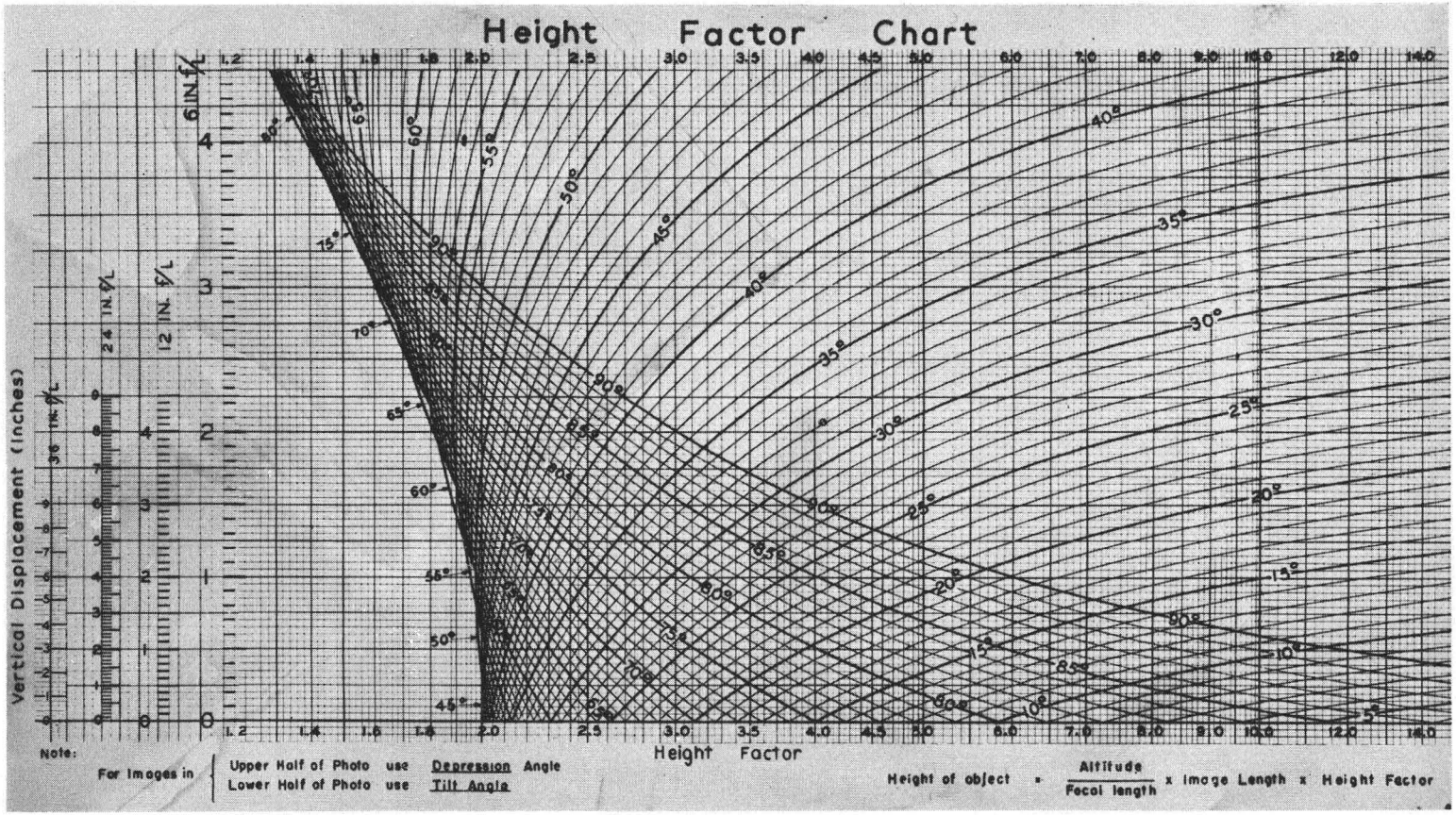

FIG. 1. Height Factor Chart—for measuring heights from small photographic images.

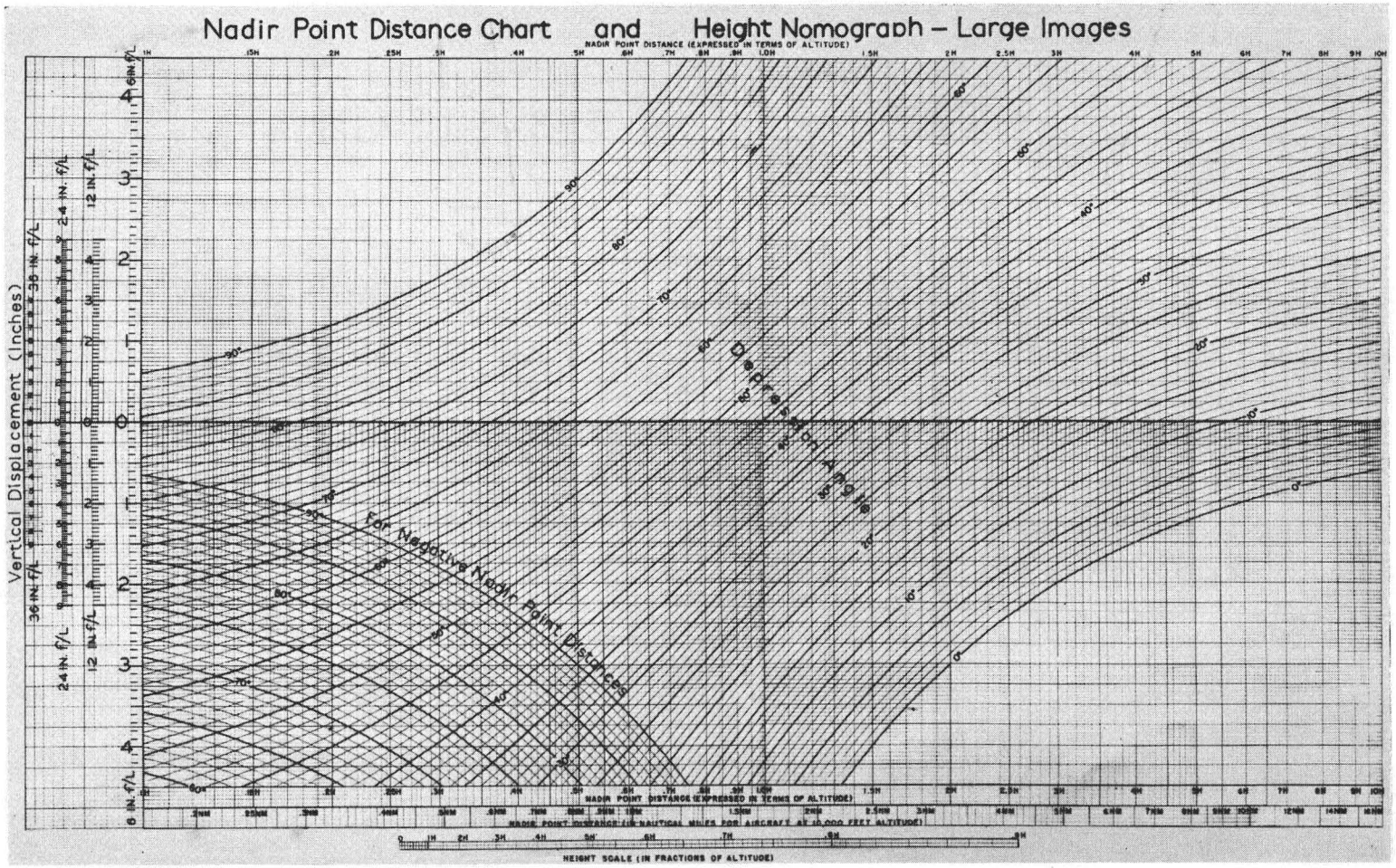

FIG. 2. Height Nomograph for measuring heights from large photographic images. Also used for determining component of nadir point distance parallel to principal line.

images the component of the image parallel to the principal line must be measured rather than the image itself. This is no great problem and may be performed quite accurately with a tube magnifier by placing the scribed measuring lines parallel to the x-axis.

As the height of the object being measured approaches an appreciable fraction of the aircraft altitude, more and more error results from use of the height-factor chart. Another nomograph was developed to overcome the error and to take full advantage of the high degree of accuracy possible with longer images.

III. Measuring Heights from Long Images

While the height method for short images resembles methods for measuring level-lying dimensions, the height method for long images is quite similar to the measurement of perpendicular heights from vertical photographs. Derivation of the two is, in fact, almost identical.

The nomograph used in measurement of heights from long photographic images is shown in Figure 2. After performing the necessary operations on the chart, the operator arrives at the height as a certain fraction of the aircraft altitude. The only mathematics necessary, then, is the multiplication of this fraction and the altitude. In this respect the method is even simpler than the corresponding technique for vertical photography, as a graphical step replaces a mathematical one.

To use the *height-nomograph—large images*, the vertical displacements to the top and bottom of the image are determined on the photograph and are applied to the chart in the same manner as previously explained for the height-factor chart, comments concerning other focal-lengths being equally applicable. However, on the height-nomograph—large images, both positive and negative vertical-displacement is presented, and the photographic depression angle is always used—never the complementary tilt-angle. Horizontal lines are extended from the two vertical-displacement values until they both intersect the correct depression-angle curve for the photograph (see Figure 3a). The horizontal distance, L, between these two intersections is then measured, or preferably taken with dividers, and is applied to the left side of the scale at the very bottom of the chart, where the height of the object being measured is given as a fraction of the altitude.

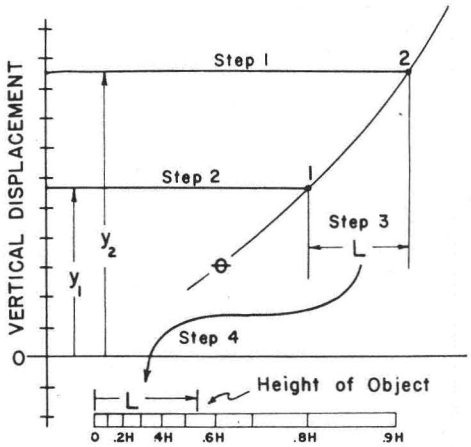

FIG. 3a. Basic technique for determining heights with height nomograph—large images.

It can readily be appreciated that measurements are more accurate the longer the image-length, but even for small images careful work should give quite accurate results. However, heights of less than about one-tenth the altitude are probably more accurately measured on the height-factor chart. This generally limits measurements from the height-nomograph—large images to lower altitude photography. For the sake of accuracy it should be mentioned that the aircraft altitude used is always the height above the *base* of the object being measured.

An alternate technique for applying the vertical-displacements of the top and bottom of the image to the chart is shown in Figure 3b. This technique provides a measure of what the author has been striving to do with all his nomographic methods—that is, to make the measurement of objects from oblique photographs as simple and as similar to measurement from vertical photographs as is possible, (without requiring the use of an electronic computer), thereby speeding up the measurements and limiting the amount of new information which a photo interpreter must learn—new information in most cases simply being the manipulation of the nomographs.

In this technique for applying the vertical-displacements, the displacement to the top of the image is first taken with dividers, or proportional dividers, and the

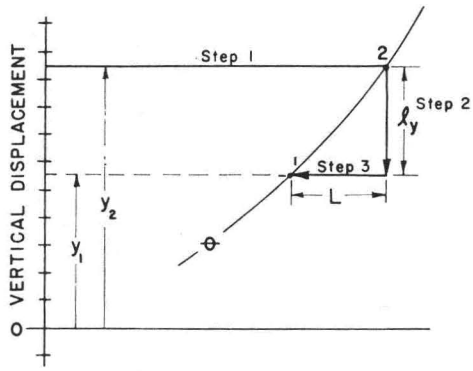

FIG. 3b. Alternate technique for determining distance, L, on height nomograph—large images.

proper depression-angle intersection is located on the chart and marked. Then, as with vertical photography, the image-length, or in this case, the difference in vertical-displacements, is taken with the dividers and is laid off vertically downward from the marked point. Dividers are next rotated about the bottom point, and the distance, L, to the depression-angle curve is taken for application to the height scale.

On the height-nomograph—large images depression-angle curves have been plotted for every degree from 0 to 90. It can be seen that application to the chart of depression-angle values measured to within a fraction of a degree is less simple and not as accurate as for whole degrees. Therefore, it is suggested that the depression-angle value be rounded off to the nearest whole degree; or if necessary, two measurements may be made for the curve either side of the correct depression angle, and the final answer weighted accordingly.

Although not specifically developed for determining nadir-point distances, the height-nomograph—large images serves an important auxiliary function for this use. The nadir-point distance is the distance of a point on the ground from a point directly underneath the camera at the time of exposure. To fully define the position of the ground point, it is necessary to know both the location of the nadir and the azimuth of the point from the nadir with respect to some known azimuth. The height-nomograph merely gives the component of the nadir point distance parallel to the camera axis, or on the photograph, parallel to the principal line. It is determined on the chart at the proper intersection of (1) vertical-displacement of the image point and (2) photographic depression-angle by the scales at top and bottom of the chart in fractions and multiples of the altitude. There is an additional conversion scale at the bottom for determining this component in nautical miles for an aircraft at 10,000 feet altitude, which may be quickly converted for other altitudes by multiplying the ratio of actual altitude to 10,000 feet.

The component of nadir-point distance perpendicular to the principal line may be determined nomographically by previously mentioned methods developed by the writer. The ground point may then be located, provided the position of the nadir and the azimuth of the photographic axis is known.

The over-riding family of curves in the lower left of the height-nomograph—large images is used for measuring negative nadir-point distances and for determining heights from images lying below the nadir on the photograph. The method used is exactly the same as for points and images lying above the nadir.

IV. Derivation of Mathematics and Construction of Nomographs

Derivation of the mathematical formulas required for construction of the nomographic-charts used in measuring vertical heights from oblique photographs is nearly as simple as the use of the charts themselves. The derivation of the formula on which is based the height-nomograph—large images will be explained first. All mathematics occurs on the *principal plane*—the plane which includes the principal ray, or camera axis, and the nadir-point (refer to Figure 4.).

The five independent parameters necessary to measure a vertical height, h, from a large image are as follows:

Photographic parameters

H—aircraft, or camera, altitude above base of object.
f—focal-length of camera
t—tilt-angle of camera, the complement of depression-angle, θ.

Image parameters

y_1—vertical-displacement to bottom of image.

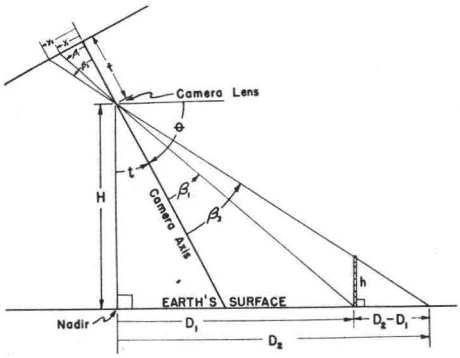

FIG. 4. Side view of oblique photo geometry for measuring vertical heights.

y_2—vertical-displacement to top of image.

Dependent parameters used in the derivation are D_1 and D_2, the nadir-point distances (on principal plane) for image-points representing the bottom and top of the object, and β_1 and β_2, the interior camera-angles for these two points. The nadir-point distances aid in the derivation in the same manner as with the corresponding derivation for vertical photography.

From the diagram (Figure 4) it is seen by similar triangles that

$$\frac{h}{H} = \frac{D_2 - D_1}{D_2},$$

and therefore,

$$h = H\left(1 - \frac{D_1}{D_2}\right). \tag{3}$$

Also it is seen that

$$D_k = H \tan (t + \beta_k), \tag{4}$$

where k is any point. Therefore, combining (3) and (4) and using β's to top and bottom of image

$$h = H\left[1 - \frac{\tan (t + \beta_1)}{\tan (t + \beta_2)}\right]. \tag{5}$$

This, very simply, is the formula utilized in the measurement of vertical heights from large images.

The height-factor chart for measuring heights from small images is based on a formula equivalent to (5) but derived by an independent method utilizing a helping plane through the base of the image perpendicular to the camera axis. The formula is:

$$h = \frac{H}{f}\, l_y \frac{\cos \beta_1 \cos \beta_2}{\cos (t + \beta_1) \sin (t + \beta_2)}, \tag{6}$$

where $l_y = y_2 - y_1$, the component of the image parallel to the principal line of the photograph. Rather than present here the derivation for (6) it is simpler to merely show its equivalence to formula (5).

From (5), substituting sine and cosine values for the tangent, and simplifying:

$$h = H\left[\frac{\cos (t + \beta_1) \sin (t + \beta_2) - \sin (t + \beta_1) \cos (t + \beta_2)}{\cos (t + \beta_1) \sin (t + \beta_2)}\right].$$

Using sum of angles reductions for the numerator, multiplying, and gathering terms:

$$h = H\left[\frac{\cos \beta_1 \sin \beta_2 - \cos \beta_2 \sin \beta_1}{\cos (t + \beta_1) \sin (t + \beta_2)}\right].$$

Again changing the form of the numerator:

$$h = H\, \frac{(\tan \beta_2 - \tan \beta_1) \cos \beta_1 \cos \beta_2}{\cos (t + \beta_1) \sin (t + \beta_2)}.$$

From Figure 4 it may be seen that,

$$\tan \beta_2 - \tan \beta_1 = \frac{y_2 - y_1}{f} = \frac{l_y}{f}.$$

Therefore, substituting:

$$h = H\, \frac{l_y}{f}\, \frac{\cos \beta_1 \cos \beta_2}{\cos (t + \beta_1)(\sin t + \beta_2)}.$$

This formula is identical to (6), thus proving the equivalence of (5) and (6).

The difference in the methods for measuring heights from long and short images is provided by a simplifying assumption made with formula (6) which cannot be made with (5). This assumption is that as the image becomes smaller and approaches zero length, β_1 and β_2 approach a common value β_0. Therefore, for images of zero length the common value β_0 may be substituted in (6) to give

$$h = \frac{H}{f}\, l_y \frac{\cos^2 \beta_0}{\cos (t + \beta_0) \sin (t + \beta_0)}. \tag{7}$$

Substitution of the common value β_0 in (5) gives the result, $h = 0$, but in formula (7), $h = 0$, only when $l_y = 0$. Therefore, (7) may be used when $l_y \neq 0$, by assuming the value of β_0 to occupy a point midway between β_1 and β_2, i.e., the image midpoint. Detailed studies have not been performed as to the error involved in this assumption, but preliminary calculations show that the error is very small when measuring heights of less than about one

tenth the aircraft altitude, increasing, however, with increasing nadir-point distance of the height. Image parameters for measuring heights from small images are, therefore, l_y and y_0, rather than y_1 and y_2.

Formula (7), it may be seen, is equivalent to formula (2) where:

$$\text{height factor} = \frac{\cos^2 \beta_0}{\cos(t+\beta_0)\sin(t+\beta_0)} \cdot \quad (8)$$

From Figure 4,

$$\beta_0 = \tan^{-1} \frac{y_0}{f} \quad (9)$$

The above two formulas, (8) and (9), are the equations for the family of curves on the height-factor chart. The vertical axis of the chart is the y_0/f axis; the horizontal axis is the height-factor axis. The latter values are plotted logarithmically; the former, directly for direct application of vertical-displacement values with rule or dividers. Curves are plotted for values of t from 0° to 90°.

After construction of the chart was begun, it was discovered that the curves were mirror images about the height-factor axis, and that curves for values of depression-angle above the axis corresponded to those for tilt-angle below the axis. Therefore, the bottom half of the nomograph was eliminated, the top half sufficing for both.

Explanation of the construction of the height-nomograph—large-images has been deferred to the last to show the reader tricks which are possible with nomographs, and which although at first appearing complicated, are actually quite elementary. The following solution, it may be added, was not arrived at initially by deduction, but through trial and error.

The formula on which the height-nomograph–large images is based is (5) where,

$$\text{altitude fraction} = 1 - \frac{\tan(t+\beta_1)}{\tan(t+\beta_2)} \cdot \quad (10)$$

The formula for the family of curves on the nomograph is simply

$$m_k = \tan(\pm\beta_k), \quad (11)$$

where

$$\beta_k = \tan^{-1} \frac{y_k}{f}, \quad (12)$$

k being any point. The vertical axis, as with the height-factor chart, is the y_k/f axis; the horizontal, the m_k axis. Curves are plotted for whole number values of t from 0° to 90° corresponding on the chart to depression-angle value of 90° to 0°. The y_k/f values are plotted directly; m_k values, logarithmically.

The m_k values plotted on the logarithmic scale act similarly to a slide rule. A given distance along the scale represents the difference in logarithms of the numbers at the ends of the given distance. The number, or antilogarithm, corresponding to the given distance is the quotient of the two numbers. Thus, the distance, L, in Figure 3 actually represents the quotient:

$$\frac{m_1}{m_2} = \frac{\tan(t+\beta_1)}{\tan(t+\beta_2)} \cdot \quad (13)$$

This distance applied to the proper scale would give the value of (13) which, when subtracted from unity (10), is the proper altitude fraction. However, by going one step further, the values on the scale at the bottom of the height nomograph—large images have already been subtracted from unity. Application of the distance L to the scale thus solves (10) completely, giving the height-measurement directly as a fraction of the altitude.

The mathematics for the component of the nadir-point distance lying on the principal-plane has already been derived—formula (4). And by comparing (4) and (11), the latter being the equation for the curves on the height-nomograph—large images, it is seen that these curves give the direct solution for nadir-point distance

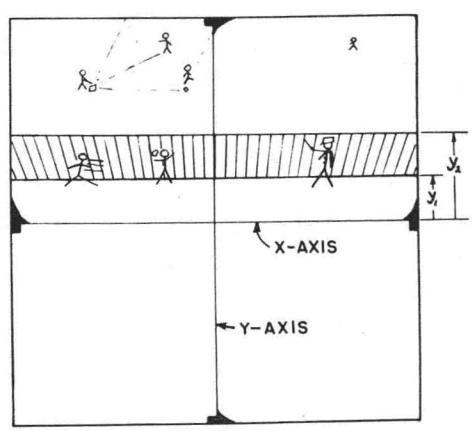

FIG. 5. Vertical displacements, y_1 and y_2, are the same for all boards in fence, proving validity of methods for performing measurements of vertical heights from images located anywhere on the photograph.

in one operation. Thus, it was only necessary to add the proper scales for nadir-point distance along the horizontal axis.

So far in this discussion, the mathematics has not departed from the principal plane, and explanations for the nomographs have blandly implied that the method is valid for any point on the oblique photograph. A non-rigorous, but lucid, proof of the validity of the mathematics for any height image on the photograph is here presented.

In Figure 5 is shown a schematic diagram of an oblique photograph containing a board fence parallel to the x-axis of the photograph. All the boards in the fence are vertical and of equal height, and on the photograph they all project identical vertical displacements. Therefore, the foregoing mathematics, which was derived only for the center board, i.e., the board lying on the principal line, will give an identical height for all the boards, *provided* the vertical displacements, or component of the image parallel to the y-axis, is applied to the formulas or nomographs.

V. Acknowledgments

The writer would like to acknowledge the aid and encouragement given by Major Byron L. Schatzley, USAF, who was responsible for the establishment and supervision of this project at Wright Air Development Center, Wright Patterson Air Force Base, Ohio, and would also like to thank personnel of the Computer Laboratory, WADC, for computation of points for the nomographs.

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news note

Special Cameras are Being Tested

Special automatic cameras which make a record of a radar screen are being experimentally put into operation by the Civil Aeronautics Administration to evaluate the Air Traffic Control Radar Beacon System.

Use of the special cameras manufactured by Gordon Enterprises, North Hollywood (Calif.) at C.A.A. radar sites in New York, Chicago, Washington and Norfolk has been announced by D. M. Stuart, Director of the C.A.A. Technical Development Center, Indianapolis.

The move may presage use of such cameras to record almost all aircraft landings and take-offs. Extension to marine use might in the future provide definite records of responsibility for such disasters as the Andrea Doria-Stockholm collision which occurred off Nantucket on July 25.

The special cameras, made to C.A.A. specifications, will record information on plan-position radar indicators during special flight tests or during periods in which it is desired to record actual movement of air traffic along airways or in traffic patterns near airports, Stuart said.

"When the information is recorded on film, it is possible to make detailed studies after the test period has been completed to be sure that no significant data has been overlooked," Stuart said.

The Gordon Enterprises cameras, using 35 mm film in 100 foot magazines, are operated automatically by the sweep of the radar antenna. In addition to photographing the radar screen, the camera simultaneously photographs a clock face, so that the exact time is included as part of each picture. A data card for pertinent information is also photographed.

One feature of the cameras is that they can constantly photograph the radar screen unattended by an operator. Except for changing of film and winding of the timepiece, the cameras will perform in excess of 500 hours without repair or adjustment.