*The Effects of Earth's Curvature and Refraction on the Mensuration of Vertical Photographs**

J. W. SMITH,
N. Y. College of Forestry,
Syracuse University, Syracuse, N. Y.

COMPARED to errors from other sources, the errors introduced into most conventional photogrammetric techniques by earth's curvature and atmospheric refraction are of such small magnitude as to be negligible. However, analytical control extension, for relatively long strips of photographs, is made less tedious and more economically feasible by high-speed electronic computing methods. Accordingly, such factors as earth's curvature, atmospheric refraction and meridian convergence require consideration. Utilization of the geocentric coordinate system, devised by Professor Earl Church [1] in 1951 and more recently described by Dr. C. S. Shu and Professor A. J. McNair, [3] effectively eliminates errors due to earth's curvature, and compensates for the effect of meridian convergence.

Because of the longer image rays and the larger areas covered, it might be supposed that high-altitude photography would be subject to significantly greater errors from earth's curvature and atmospheric refraction. For example, a photograph taken at an altitude of 15,000 feet by a $9'' \times 9''$ format camera with a $6''$ focal-length lens covers approximately 18 square miles; if the same camera is used at 50,000 feet, the area covered by each photograph is approximately 200 square miles. From this observation it may logically be concluded that if there is willingness to expend the same energy areawise, eleven times as much effort, either instrumentally or analytically, may be concentrated on the high-altitude photograph. The small scale and resulting loss of detail are likely to cancel this superficial advantage and to restrict the use of high-altitude photography, for the present at least, to the military.

The purpose of this paper is not, however, to analyze or even discuss the merits of one or another type of photography. Rather, it is an investigation into the nature and magnitude of curvature and refraction errors as they may be expected to occur in aerial photographs. Also included are some suggestions for the analytical elimination or compensation of such errors when the number of photographs being dealt with does not justify the use of geocentric coordinates.

CURVATURE

Figure 1 illustrates the relationships, greatly exaggerated, between a vertical aerial photograph and the earth's surface. In which

L = exposure station
f = focal-length of lens
o = principal point of the vertical photograph
e = photographic image of ground-point C
$oe = m$ = radial distance to point e
H = flying height above a sea-level datum, assumed coincident with the earth's surface
V = ground nadir point
E' = intersection of ray LeC with a horizontal plane through V
$VE' = M$
V_1 = interesection of LV extended and a horizontal plane through C
$V_1C = M_1$
$VV_1 = DC = h_c$ = a correction to scale data or flying height for earth's curvature.
$OV = OC = R$ = radius of earth (mean value of semi-major and semi-minor

* In the 1957 contest for the Bausch & Lamb Photogrammetric Award, this paper was awarded one of the two Second Prizes.

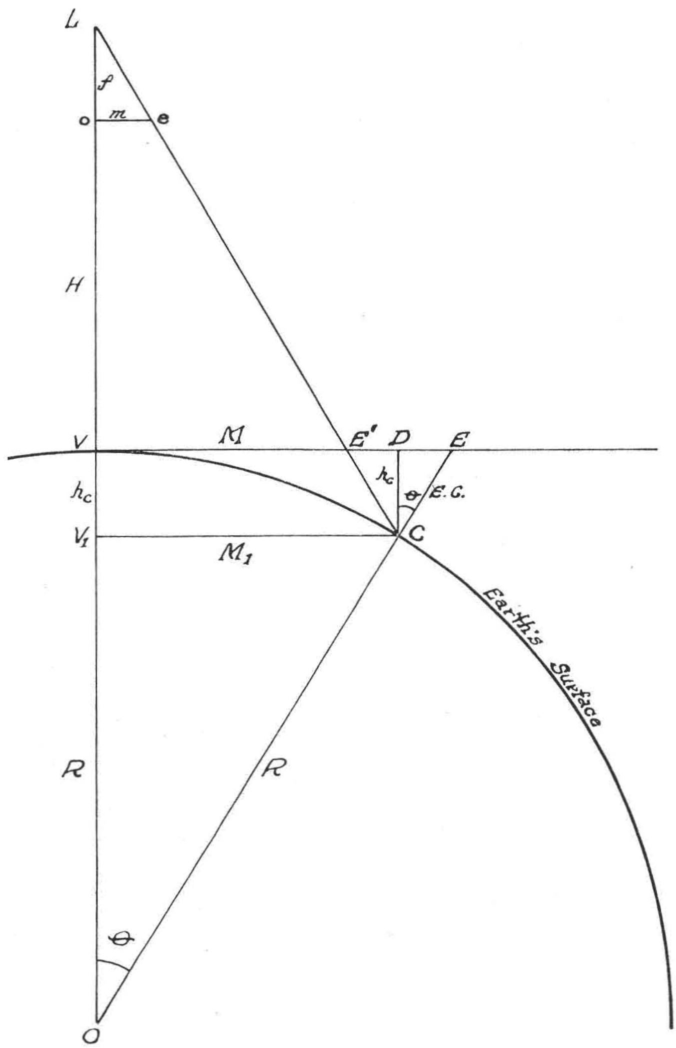

FIG. 1. Error caused by earth's curvature.

axes of Clark's 1866 spheroid is 20.888×10^6 feet)

O = center of the earth

θ = angle subtended at the earth's center by arc VC

$EC = E.C.$ = elevation error.

From the diagram, it is evident that:

$$R^2 = (R - h_c)^2 + M_1^2 \tag{1}$$

Expanding:

$$R^2 = R^2 - 2Rh_c + h_c^2 + M_1^2 \tag{2}$$

$$h_c^2 - 2Rh_c + M_1^2 = 0 \tag{3}$$

To circumvent the necessity of a quadratic solution the assumption can be made that h_c^2 is a negligible quantity, as indeed it is when compared to $2Rh_c$ and M_1^2.

Equation (3) then becomes:

$$h_c = M_1^2/2R \tag{4}$$

Taking the first derivative of (4):

$$dh_c = 2M_1/2R \cdot dM_1 \tag{5}$$

Reducing:

$$dh_c = M_1/R \cdot dM_1 \tag{6}$$

Substituting the value of R in (6):

$$dh_c = M_1/20.88 \times 10^6 \cdot dM_1 \tag{7}$$

From equation (7) it is evident that there must be a considerable change in the value of M_1 to effect a small change in the computed value of h_c. From this observation it is concluded that the computed value of h_c will be sufficiently precise if the assumption

TABLE 1

Radial distance (m) in inches from photograph	h_c *in feet*
4.5″	33.7′
4.0″	26.5′
3.5″	20.4′
3.0″	15.0′
2.5″	10.4′
2.0″	6.7′
1.5″	3.7′
1.0″	1.7′
0.5″	0.4′
0.0″	0.0′

is made that M is equal to M_1.

Then, substituting M for M_1 in equation (4):

$$h_c = M^2/2R \tag{8}$$

It is observed in the diagram that R is very nearly equal to $(R-h_c)$, and as a consequence the cosine θ is almost unity, and θ is an extremely small angle. This leads to the observation that $E.C.$ is approximately equal to h_c. The computed value for h_c can then be considered equal to the height error, $E.C.$, for point C caused by the curvature of the earth.

Since $M=H/f\cdot m$, equation (8) may be rewritten:

$$h_c = (H/f)^2\cdot(m)^2/2R \tag{9}$$

If a camera focal-length of 6″ is assumed and a value of 50,000 feet for the flying height (H), Table 1 gives the relationship between various radial distances corresponding to m and the computed value of h_c.

The information and relationship shown in Table 1 are further illustrated in Figure 2.

The following equation, in addition to those already mentioned, is evident from Figure 1:

$$M_1 = (H + h_c)/f\cdot m$$

The computed distance M_1 may be assumed equal to arc distance VC for any values with which there is concern. That this is true can be seen from the following equations:

$$\sin\theta = M_1/R \tag{11}$$

Since θ is a very small angle:

$$\theta'' = M_1/R \sin 1'' \tag{12}$$

This θ is converted to degrees by dividing by 3,600″/degree. θ may then be converted to radians by factoring by $\pi/180$ giving:

$$\theta = M_1\pi/R \sin 1''\cdot(3{,}600)\cdot 180 \tag{13}$$

Reducing, assuming the arc 1″ equals 4.848×10^{-6}

$$\theta = M_1\pi/R3.141504 \tag{14}$$

$$\theta_{\text{radians}} = \text{Arc } VC/R \tag{15}$$

Equating the values of θ given by (14) and (15):

$$\text{Arc } VC = M_1\pi/3.141504 \tag{16}$$

The value in the denominator of (16) is so nearly equal to the value of π, that for all except large values of VC or M_1, the assumption that they are equal results in a negligible error.

It can be seen from Figure 1 that if it is wished to establish the relative ground-coordinates of point C from the photo-coordinates of its corresponding image point e, the procedure should be as follows:

Assuming, as aforementioned, that the photograph is truly vertical and that, for the moment, point C lies on the horizontal datum:

Letting X_c and Y_c and x_e and y_e equal the ground-coordinates and photo-coordinates of point C and its image point e, respectively, the relationships between them,

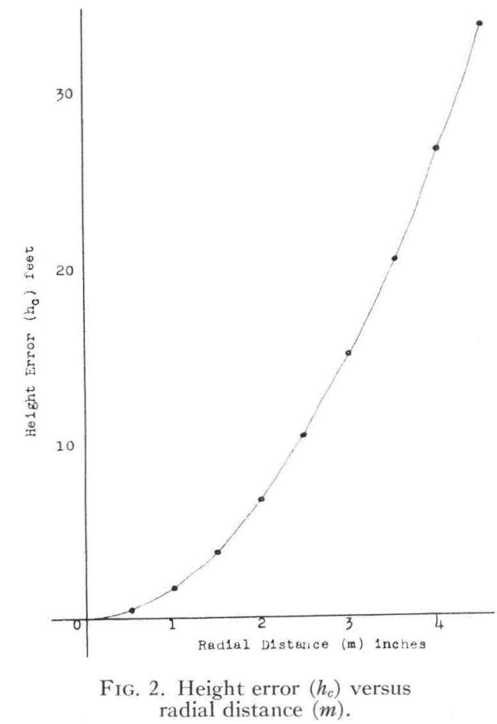

FIG. 2. Height error (h_c) versus radial distance (m).

neglecting the curvature of the earth (i.e. assuming that a horizontal plane through V is coincident with the earth's surface), are:

$$X_c = H/f \cdot x_c \quad \text{and} \quad Y_c = H/f \cdot y_c \qquad (17)$$

As is usually the case, point C does not usually lie on the datum plane, and if the elevation of point C above the datum is given by h, formulas (17) are modified to compensate for this so that:

$$X_c = (H - h)/f \cdot x_c \text{ and } Y_c = (H - h)/f \cdot y_c \quad (18)$$

From the foregoing, it is evident that to correct for the error due to earth's curvature, there may be introduced another term (viz. h_c), further modifying (18) to:

$$X_c = (H - h + h_c)/f \cdot x_c \text{ and}$$
$$Y_c = (H - h + h_c)/f \cdot y_c \qquad (19)$$

Of course, factoring the rectangular photo-coordinates by any factor is exactly equivalent to factoring the radial distance from the origin to the point, since:

$$m^2 = x_c^2 + y_c^2 \text{ and } m = \sqrt{x_c^2 + y_c^2} \quad (20)$$

If m is factored by some factor F:

$$mF = \sqrt{(Fx_c)^2 + (Fy_c)^2} \qquad (21)$$

Thus, the error due to earth's curvature might be considered a radial displacement toward the center of the photograph. Referring to Figure 3, it is apparent that the photographic image displacement ee' is caused by the curvature of the earth. The displacement ee' is derived as follows:

$$ee' = oe' - oe \qquad (22)$$
$$oe' = f/H \cdot M \qquad (23)$$
$$oe = f/(H + h_c) \cdot M \qquad (24)$$

Substituting (23) and (24) in (22)

$$ee' = Mfh_c/H(H + h_c) \qquad (25)$$

It is intuitively and mathematically apparent that the magnitude of this displacement ee' is directly proportional to the radial distance oe of the image point e from the principal point of the photograph. Since oe is directly proportional to H and inversely proportional to f, it follows that, other things being constant, an increase in the flying height (H) or a decrease in the focal length (f) will result in a greater apparent image displacement due to earth's curvature. For any given combination of f, H, M, and computed h_c, the error ee' may be computed from formula (25) and if it is of measurable magnitude, correction may be made.

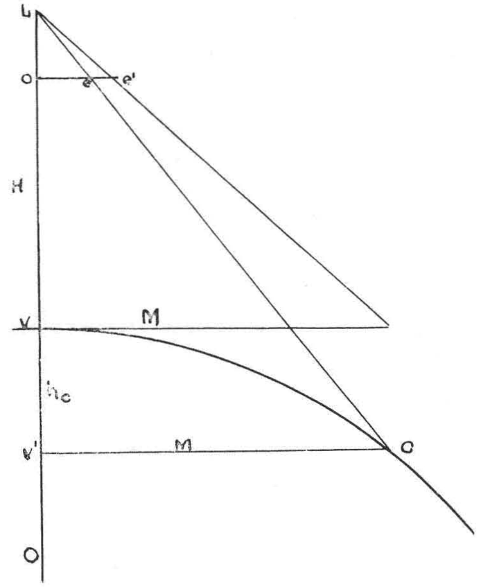

FIG. 3. Apparent photographic image displacement caused by earth's curvature.

REFRACTION

Figure 4 illustrates the image displacement on an aerial photograph caused by atmospheric refraction. In which

L = exposure station
f = focal-length
o = principal-point of photograph
V = ground nadir-point
A and B = two ground-points
b = apparent image registration on the photograph of point A on the ground
$e = ab$ = image displacement on photograph caused by refraction
R = angle of refraction or the angle between the tangent LbB to the refracted ray and LaA the ray position if there were no refraction
I = the difference in the inclination of a tangent to the ray at A and a tangent to the ray at L, therefore the change in inclination of the ray due to refraction (N.B. $I/2 = R$)
H = elevation of exposure station above the datum, represented by VAB.

The exact determination of the amount of atmospheric refraction corresponding to any altitude is a problem that cannot be completely solved, since the refraction depends on the temperature and density of the air, not only at the surface of the earth, but

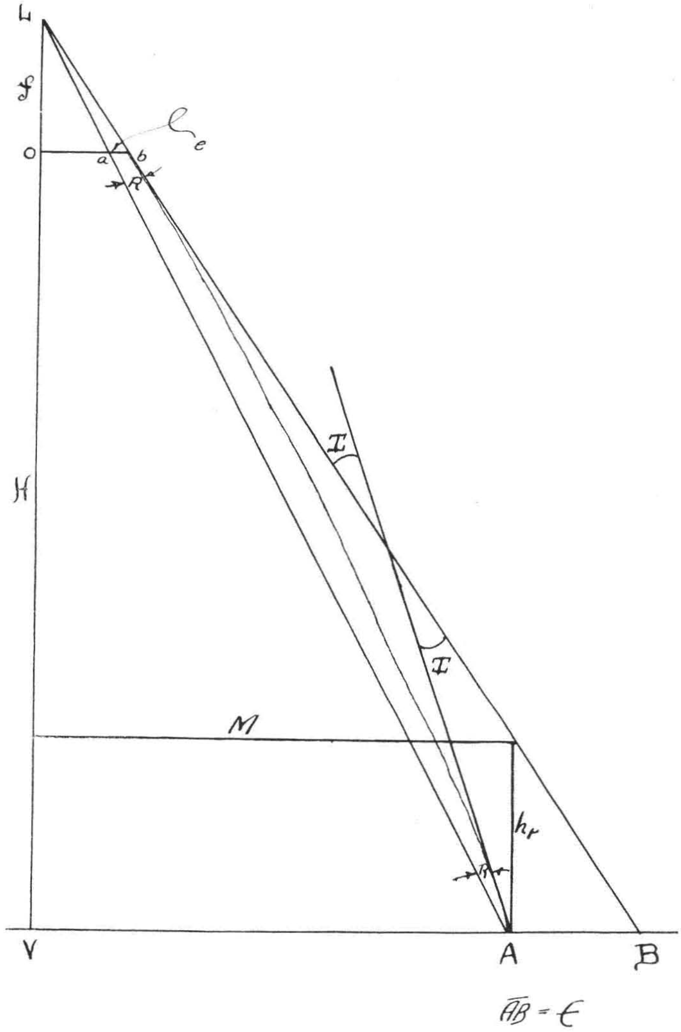

FIG. 4. Apparent photographic image displacement on a vertical photograph caused by atmospheric refraction.

also along the whole path of the ray.

Examination of Bessel's table of refractions for astronomic use indicate that atmospheric refraction for the ray inclinations encountered in photogrammetric techniques using vertical aerial photographs is so small as to be negligible. For example:

$$\tan oLb = ob/f \tag{26}$$

$$\tan oLa = oa/f \tag{27}$$

$$ab = e = f(\tan oLb - \tan oLa) \tag{28}$$

Since

$$oLa = oLb - R, \tag{29}$$

$$e = f(\tan oLb - \tan (oLb - R)) \tag{30}$$

It is apparent on the photograph that point A on the ground which should register at point a on the photograph actually registers at point b, thus being radially displaced by an amount e. By introducing a scale factor the correct radial ground distance VA can be computed from the radial photo distance ob from the following formula:

$$M = (H - h_r)/f \cdot ob \tag{31}$$

As before indicated, the refraction correction may be considered negligible, but if it does in some unusual circumstances warrant consideration, values of h_r can be computed and tabulated for various values of

f, H, ob, and R. So calculated, it may be inserted as a scale factor in formula 19, thus correcting for earth's curvature and atmospheric refraction simultaneously. Equations (19) then become:

$$X_c = (H - h + h_c - h_r)/f \cdot x_c$$
$$Y_c = (H - h + h_c - h_r)/f \cdot y_c \qquad (32)$$

It is apparent that the effect of atmospheric refraction is to radially displace the image-point from the principal-point of the vertical photograph.

Conclusions

Apparently the effect of atmospheric refraction is to partially offset the effect of earth's curvature. It is, however, nearly negligible, and difficult to accurately evaluate quantitatively.

Because of the difficulty of theoretically evaluating the effect of atmospheric refraction, the possibility of making empirical determinations of atmospheric refraction and earth curvature in combination suggests using balloon photography over an area with sufficient control to permit the precise determination of image point displacements incidental to photography taken at various altitudes with various types of cameras.

The possibility of correcting for earth's curvature and refraction by equations (32) on isolated photographs taken at high altitudes appears to be practical, since the relative ground coordinates so established can be used to determine scalar quantities and areas. The effect of tilt on this technique remains to be investigated.

Bibliography

[1]. Church, Earl, "Coordinate Determination of Ground Points on Aerial Photographs," (Air Force Problem 48), Problem Report No. 25, June 15, 1951.

[2]. Kelsey, Ray A., "Effects of Earth Curvature and Atmospheric Refraction upon Measurements Made in Stereoscopic Models of High Altitude Photography," Project 8-35-08-001, Engineer Research and Development Laboratories, Report 1242, July 9, 1952.

[3]. Shu, C. S. and McNair, A. J., "Determination of Geographic Coordinates, Flight Heights, and True Orientation for Extensions of Strips of Aerial Photographs," PHOTOGRAMMETRIC ENGINEERING, September 1956, Volume XXII, No. 4, pp. 637–643.

*Photogrammetric Applications of Radar-Scope Photographs**

PAMELA R. HOFFMAN,
Aeronautical Chart & Information Center,
St. Louis, Missouri

ABSTRACT: *The utilization of radar-scope photographs as standard photogrammetric source material has been studied for over ten years. Almost the entire bulk of completed work is experimental in nature but has provided valuable information. It has led to the development of special compilation procedures, and special compilation equipment, as well as the capability of applying these developments in production compilation. Improvement of the basic radar equipment will permit even greater success in this field in the future. This paper describes procedures and techniques for applying a specialized kind of photogrammetric knowledge in extracting topographic information from radar-scope photographs.*

* Presented at the Society's 24th Annual Meeting, Hotel Shoreham, Washington, D. C., March 27, 1958.