where

$x_{1 \text{ or } 2}$, $y_{1 \text{ or } 2}$ = photographic coordinates
X, Y, Z = ground-coordinates
X_n, Y_n = ground-radii-coordinates
f = camera focal-length
$H^{1 \text{ or } 2}$ = flying altitude
$A_{ij}^{1 \text{ or } 2}$ = orientation matrix element involving Eulerian angles at the exposure station 1 or 2

Solving for x and y in normal algebraic form,

$$x_{1 \text{ or } 2} = f\left[\frac{A_{11}^{1 \text{ or } 2}(X - X_n) + A_{21}^{1 \text{ or } 2}(Y - Y_n) - A_{31}^{1 \text{ or } 2}(H^{1 \text{ or } 2} - Z)}{-A_{13}^{1 \text{ or } 2}(X - X_n) - A_{23}^{1 \text{ or } 2}(Y - Y_n) + A_{33}^{1 \text{ or } 2}(H^{1 \text{ or } 2} - Z)}\right] \quad \text{Eq. (1)}$$

$$y_{1 \text{ or } 2} = f\left[\frac{A_{12}^{1 \text{ or } 2}(X - X_n) + A_{22}^{1 \text{ or } 2}(Y - Y_n) - A_{32}^{1 \text{ or } 2}(H^{1 \text{ or } 2} - Z)}{-A_{13}^{1 \text{ or } 2}(X - X_n) - A_{23}^{1 \text{ or } 2}(Y - Y_n) + A_{33}^{1 \text{ or } 2}(H^{1 \text{ or } 2} - Z)}\right] \quad \text{Eq. (2)}$$

The expression for an extrapolation to an expected elevation ($Z°$) at the next profile point, based upon previously known elevations (Z^{cor}), is given by 8:

$$Z_{Y+1}{}^{o} = (1 + \alpha)Z_Y{}^{cor} - \alpha Z_{Y-1}{}^{cor} \quad \text{Eq. (3)}$$

where α is a constant ≤ 1.

Some Notes on the Displacement of Photographic Images Caused by Tilt and Relief

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ABSTRACT: *Formulas are developed to express the relationships between angles measured about the principal-point, isocenter, and nadir-point in the plane of a tilted photograph and corresponding angles on the ground. These formulas are then analyzed to determine errors of photographic directions under selected conditions of tilt and relief.*

BOTH graphical and analytical principal-point radial triangulation are based on the assumption that the aerial photographs are absolutely vertical, because angles between rays emanating from the principal-point of a vertical photograph are precisely equal to corresponding angles on the ground, regardless of ground relief. Thus, in theory, there is a direct analogy between photogrammetric radial triangulation and triangulation conducted by the usual ground-survey methods.

In actual practice, of course, the aerial photographs are tilted up to a maximum of about three degrees and, except for lens distortion, the principal-point has no geometric significance as a radial-center, because relief displacements radiate from the nadir-point of a tilted photograph and tilt displacements radiate from the isocenter. In fact, if the photographic images are displaced both because of tilt and relief, there is no single point on a photograph which can correctly be used as the radial-center. Accordingly, until some means is provided for keeping the axis of the aerial camera absolutely vertical, the accuracy of photogrammetric radial triangulation will be limited by these image displacements regardless of whether the principal-point, the nadir-point, or the isocenter is used. However, the amount of error introduced is dependent upon the radial-center and, so, in the following paragraphs, the relationships between directions from these points on the photograph and corresponding directions on the ground are established under varying conditions of tilt and relief, in order to provide a means for evaluating radial triangulation in terms of the radial-center.

DIRECTIONAL ERRORS ABOUT THE PRINCIPAL POINT OF A TILTED PHOTOGRAPH:

Figure 1 shows a tilted aerial photograph with the principal line designated as *vw*, with the principal-point at *o*, and with the image of ground-point *P* located at *p*. The angle *pow* in the plane of the photograph is designated as α and represents the angle at the principal-point between the principal line and the radial line through *p*. Line *wp* is perpendicular to the principal line an,d therefore, is horizontal. Triangle *pmw* is a horizontal right-triangle formed by projecting the principal point, *o*, along the camera axis, *Lo*, to the horizontal plane containing line *wp*. Triangle *mow* is a right-triangle in the principal-plane of the photograph with the angle at *w* being equal to the angle of tilt, *t*.

If assumed for the moment that the ground surface is perfectly flat, the true angle on the ground between the principal plane of the photograph and the ground point *P* is angle *POW*, which is designated as θ in Figure 1. The problem, then, is to determine the relation between α, the angle about the principal point of the tilted photograph and θ, the corresponding true angle on the ground.

In Figure 1, it is easily seen that horizontal right triangles *POW* and *pmw* are similar and, therefore

$$\text{angle } POW = \text{angle } pmw = \theta$$

We also see that in triangle *pmw*

$$\tan\theta = \frac{wp}{mw} \tag{1}$$

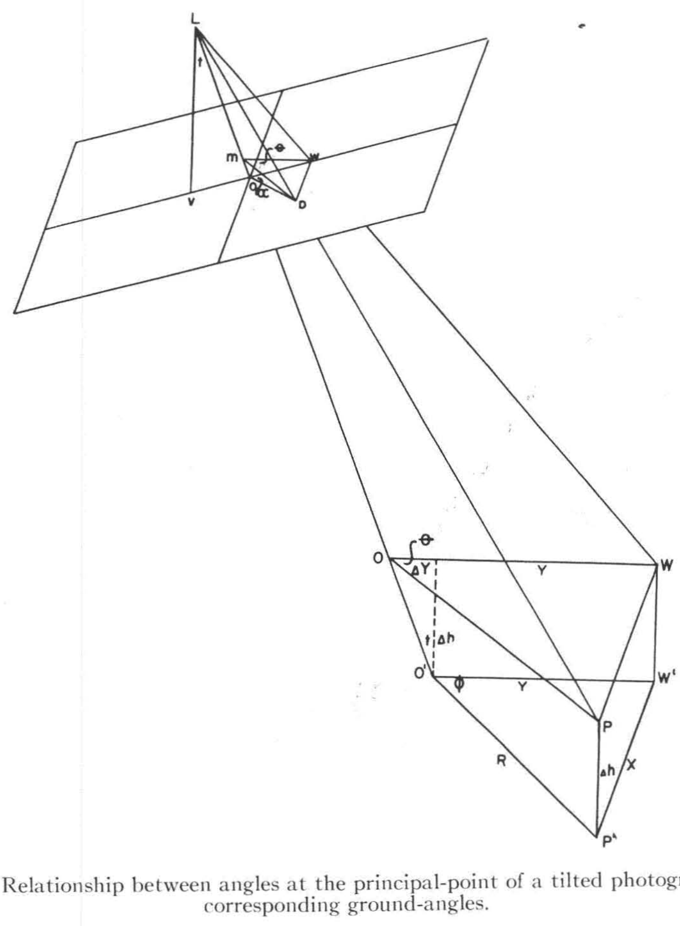

FIG. 1. Relationship between angles at the principal-point of a tilted photograph and corresponding ground-angles.

But, in right-triangle *wom*

$$mw = ow \sec t \qquad (2)$$

and, in right triangle *pow*, in the plane of the tilted photograph,

$$wp = ow \tan \alpha \qquad (3)$$

Substituting equations (2) and (3) in (1) we find that

$$\text{Tan } \alpha = \tan \theta \sec t \qquad (4)$$

The curves in Figure 2 show the difference between the true angle, θ, on the ground and the corresponding angle, α, in the plane of the photograph for selected values of tilt up to four degrees. These curves also furnish data for comparison of angles at the principal-point between any two images with corresponding ground angles. For example, if assumed that on a photograph tilted two degrees there are two images a and b so situated that the angle α between the radial-line through a and the principal-line is 20°, and the angle α' between the radial-line through b and the principal-line is 150°, then, the curve in Figure 2 for a tilt of two degrees indicates that α is in error by $+0° \; 00' \; 40''$ and that α' is in error by $-0° \; 00' \; 54''$. Consequently, the error in the angle subtended at the principal-point by images a and b, which is equal to the difference between the errors in the individual directions, is $0° \; 01' \; 34''$. It should be apparent that the maximum error in the angle subtended at the principal-point by any two images will occur when the angle is 90°, and when the angle between one radial and the principal-line is 45 degrees.

Returning now to Figure 1, it is assumed that the ground surface is not perfectly flat and that the point O' represents the point of intersection of the camera axis, Lo, with the ground surface. The difference in elevation between O' and ground-point P is indicated as Δh, and P' and W' represent the vertical projection of points P and W to the horizontal plane containing O'. The angle $P'O'W'$, designated as ϕ, represents the true horizontal angle on the ground between the principal-plane of the photograph and the point P.

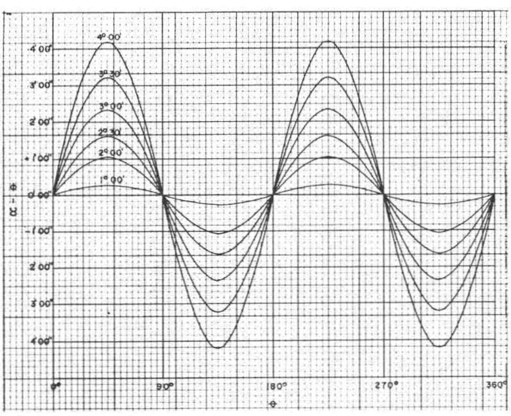

FIG. 2. Errors in radial directions at the principal-point caused by tilt (flat terrain).

In Figure 1, it is seen that

$$\tan \theta = \frac{X}{Y + aY}$$

and that

$$\Delta Y = \Delta h \tan t$$

Therefore,

$$\tan \theta = \frac{X}{Y + \Delta h \tan t} \qquad (5)$$

Dividing both the numerator and denominator or the right-hand side of (5) by R, the horizontal distance from O' to P, and substituting appropriate trigonometric functions for ϕ, we obtain

$$\tan \theta = \frac{\sin \varphi}{\cos \varphi + \frac{\Delta h}{R} \tan t} \qquad (6)$$

We may now substitute (6) in (4) and arrive at an expression for α, the angle in the plane of the tilted photograph, in terms of ϕ, the true ground-angle when relief is present Thus,

$$\tan \alpha = \frac{\text{Sin } \phi \sec t}{\cos \phi + \frac{\Delta h}{R} \tan t} \qquad (7)$$

The curves in Figures 3 and 4 show differences between directions in the plane of the photograph referred to the principal-line, and corresponding ground-angles for tilts of one degree and two degrees respectively, under selected conditions of relief as expressed by the ratio $\Delta h/R$, the slope of the line connecting ground points O' and P in Figure 1. Although at first thought ground slopes of 5% or less seem very small, greater slopes than this are actually quite uncommon except in exceedingly rough terrain. Here again, these curves may be used for comparison of angles at the principal-point between any two images with corresponding ground angles.

DIRECTIONAL ERRORS ABOUT THE NADIR POINT OF A TILTED PHOTOGRAPH:

Figure 5 is similar to Figure 1 except that in this case the angle, α, in the plane of the

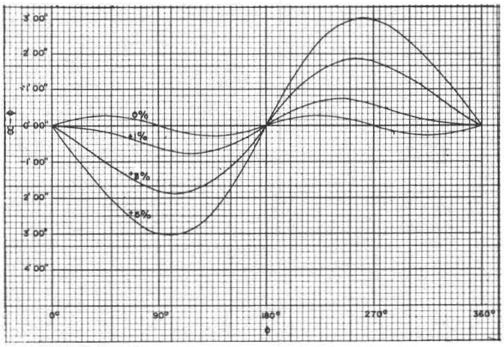

FIG. 3. Errors in radial directions at the principal-point of a photograph tilted one degree (broken terrain).

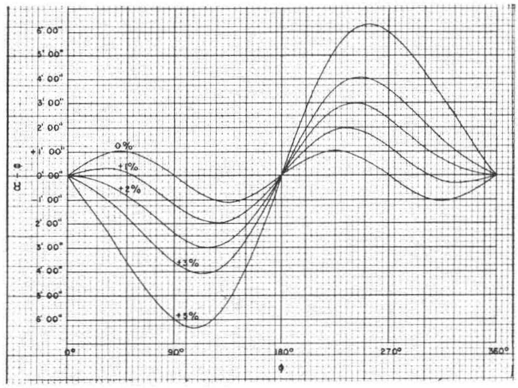

FIG. 4. Errors in radial directions at the principal-point of a photograph tilted two degrees (broken terrain).

photograph is turned about the nadir-point v, and the corresponding ground angle, θ, is turned about the ground nadir-point V, if it is assumed that V and P lie in the same horizontal plane.

It can readily be seen in Figure 5 that horizontal right-triangles pmw and PVW are similar and, therefore,

$$\text{angle } pnw = \text{angle } PVW = \theta$$

It is also apparent that

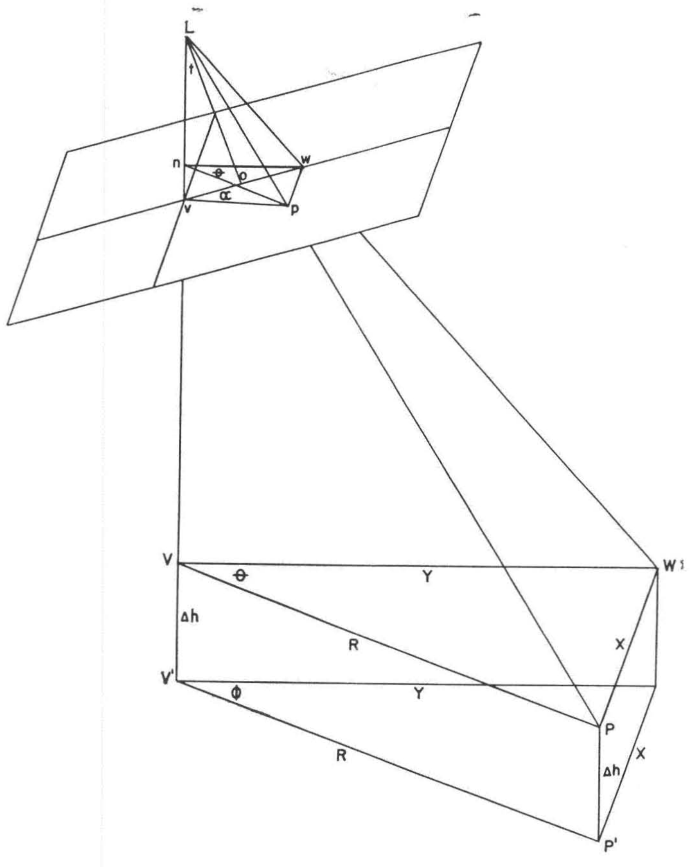

FIG. 5. Relationship between angles at the nadir-point of a tilted photograph and corresponding ground-angles.

$$\tan\theta = \frac{wp}{nw} \tag{8}$$

$$nw = vw\cos t \tag{9}$$

and that

$$wp = vw\tan\alpha \tag{10}$$

Substituting (9) and (10) in (8), we find

$$\tan\alpha = \tan\theta\cos t \tag{11}$$

which expresses the relation between α and θ when directions are turned about the nadir-point.

Comparison of equation (4) with (11) shows that these expressions are identical except that α and θ are transposed. Thus, the curves in Figure 2 are mirror images of curves showing errors in direction at the nadir point in the plane of the photograph. In other words, whereas in Figure 2 the angular error in α from 0° to 90° is positive in sign, the angular error at the nadir-point is of equal magnitude but is negative in sign.

A glance at Figure 5 is sufficient to show that even though the ground nadir-point is situated at V' at some elevation Δh below P, the angles PVW and $P'V'W'$ are exactly equal. Consequently, differences in ground elevation have no additional effect on the relation between angles measured about the nadir-point of a photograph and corresponding ground angles. Thus, it has been proved that relief displacements radiate from the nadir-point.

DIRECTIONAL ERRORS ABOUT THE ISOCENTER OF A TILTED PHOTOGRAPH:

Figure 6 is similar to Figures 1 and 5. In this case, however, the angle α in the plane of the photograph is measured about the isocenter and, under the assumption that the ground surface is perfectly flat, the corresponding ground angle, θ, is turned about point I in the horizontal plane containing point P.

In Figure 6, horizontal right-triangles pkw and PIW are similar, and so

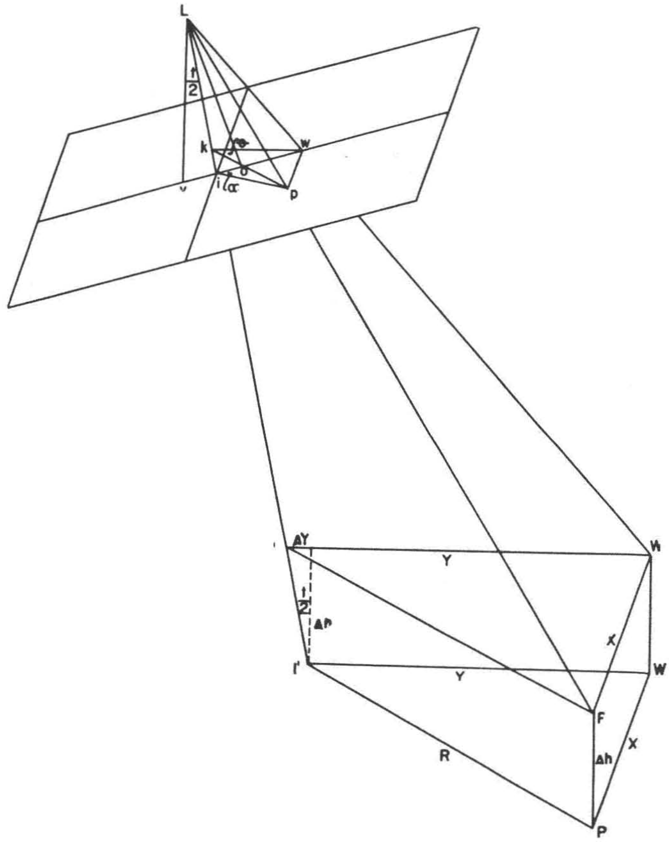

FIG. 6. Relationship between angles at the isocenter of a tilted photograph and corresponding ground-angles.

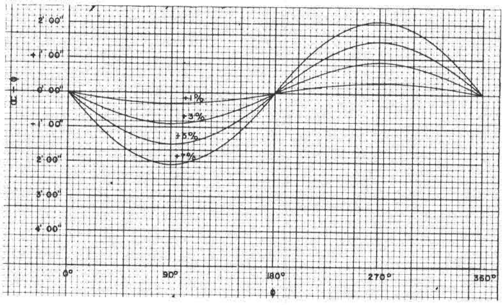

FIG. 7. Effect of relief displacement on radial directions at the isocenter of a photograph tilted one degree.

$$\text{angle } pkw = \text{angle } PIW = \theta$$

In right triangle pkw

$$\tan \theta = \frac{wp}{kw}$$

and in right triangle piw

$$\tan \alpha = \frac{wp}{iw}$$

But, triangle iwk is isosceles since

$$\text{angle } wik = \text{angle } okw = 90° - t/2$$

and so

$$iw = kw$$

Therefore,

$$\tan \alpha = \tan \theta = \frac{wp}{kw}$$

and

$$\alpha = \theta$$

It has been proved then, that tilt displacements radiate from the isocenter of a photograph by demonstrating that angles turned about the isocenter are precisely equal to corresponding ground-angles when the ground is perfectly flat.

There will now be examined the situation where the point on the ground, I', in Figure 6, corresponding to the isocenter of the photograph is at some elevation, Δh, below the ground point P. In this case, the angle $P'I'W'$ which is designated as ϕ, represents the true ground-angle.

Since already shown that

$$\alpha = \theta$$

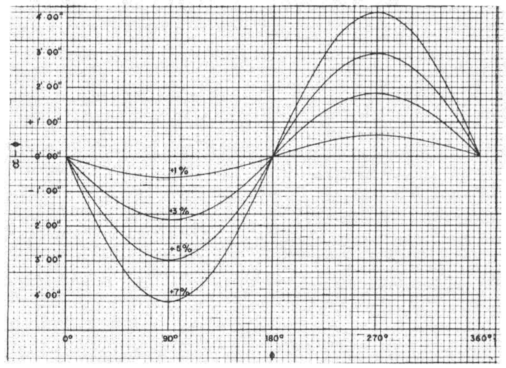

FIG. 8. Effect of relief displacement on radial directions at the isocenter of a photograph tilted two degrees.

the problem here consists merely of finding the relation between θ and ϕ. Referring to Figure 6, it is seen that

$$\tan \theta = \frac{X}{Y + \Delta Y} \qquad (12)$$

and

$$\Delta Y = \Delta h \tan t/2 \qquad (13)$$

Dividing the numerator and denominator of the right-hand side of equation (12) by R, the horizontal distance $I'P'$, and substituting appropriate trigonometric functions of ϕ,

$$\tan \alpha = \frac{\sin \varphi}{\cos \varphi + \Delta h \tan t/2} \qquad (14)$$

Figures 7 and 8 contain curves showing differences between directions in the plane of the photograph referred to the principal-line, and corresponding ground-angles for tilts of one degree and two degrees, respectively, under selected conditions of relief as expressed by the ratio $\Delta h/R$.

In conclusion, it might be well to note that the foregoing provides nothing new in a qualitative sense regarding the relative merits of the principal-point, the nadir-point, and isocenter as radial-centers for tilted photography. However, the curves of error do furnish a quantitative measure of the difficulties encountered in radical triangulation under varying conditions of tilt and relief and serve to emphasize the need for improvement in camera stabilization techniques if radial triangulation is to gain favor as a means for supplying secondary control for large scale mapping.