of geometry on flow, providing an especially clear description of supersonic and nonsteady flow phenomena. The photogrammetric techniques described herein make possible *quantitative* measurements of the absolute pressures and pressure distributions in steady or nonsteady flow fields to an accuracy compatible with water-table accuracy, although enhanced accuracy could be achieved if desired. The optical technique avoids any direct influence on the flow, and can be adapted to use automated techniques of data reduction developed for topographic mapping.

Acknowledgements

This work was made possible by support from the U. S. Naval Underwater Ordnance Station at Newport, R.I., under contract N140(122)61606B. The exploration of the photostereo method was largely performed by Alan H. MacPherson. The cooperation of the Photogrammetry Laboratory of the Civil Engineering Department, M.I.T., and Professor Charles L. Miller and Mr. Daniel R. Schurz are gratefully acknowledged. M. K. H. Chu has carried through the design and fabrication of the new cameras. Mr. Mikio Suo, Mr. James B. Frazier, Jr., and Mr. Edwin U. Sowers, III, have also importantly assisted the furtherance of the project.

Bibliography

1. H. K. Heen and R. W. Mann, "The Hydraulic Analogy Applied to Nonsteady, Two-Dimensional Flow in the Partial-Admission Turbine," *Trans. A.S.M.E.*, Vol. 83, Series D., No. 3, Sept. 1961, pp. 408–422.
2. Ippen, Harleman, and Crossley, "Studies on the Validity of the Hydraulic Analogy to Supersonic Flow," Parts I, II, III, IV, and V, U.S.A.F. Technical Report No. 5985, Hydrodynamics Laboratory, M.I.T.; May, October 1950; February, December 1952.
3. Harleman and Boedtker, "Water Table Experiment on Transient Shock Wave Diffraction," Parts I and II, Hydrodynamics Laboratory, M.I.T., Technical Reports 12, August 1953, and 13, October 1954.
4. Ippen and Harleman, "Verification of Theory for Oblique Standing Waves," *Trans. A.S.C.E.*, Vol. 121, 1956, pp. 678–694.
5. Harleman and Ippen, "The Range of Application of the Hydraulic Analogy in Transonic and Supersonic Aerodynamics," Extrait des Memoires sur la Mechanique des Fluides Offerts a M.D. Raibouchinsky a l'Occasion de son Jubile Scientifique.
6. A. H. MacPherson, "The Use of Stereophotography and the Hydraulic Analogy to Study Compressible Gas Flow in a Partial Admission Turbine," S.M. Thesis, Department of Mechanical Engineering, M.I.T., Jan. 19, 1959.
7. M. Suo, "Analysis and Experiment on Unsteady Gas Flows Using the Hydraulic Analogy," S.M. Thesis, Department of Mechanical Engineering, M.I.T., Jan. 1960.

Tissot's Indicatrix and Photogrammetry

W. R. TOBLER,
Department of Geography,
Univ. of Michigan, Ann Arbor, Mich.

In 1881 the Frenchman M. A. Tissot provided certain theorems concerning the inevitable alteration of geometric relations which occurs on plane maps of the sphere or ellipsoid.[1] Since one of the problems of photogrammetry is the determination of the amounts by which geometric relations on the photograph differ from those on the object photographed, it is felt that a short review of Tissot's work will be of interest to photogrammetrists. Tissot's results are completely general and can be applied to aerial photographs, to radar images, or to camera lenses.

The concern is with transformations which take elements of one set into elements of a second set. This can be written as a pair of equations

$$u' = f_1(u, v)$$
$$v' = f_2(u, v)$$

The u', v' are to be interpreted as being plane coordinates on a piece of film (map, image) and u, v are coordinates used to describe locations on the original two-dimensional surface. In general it is assumed that the functions f_1 and f_2 are real, single-valued, continuous and differentiable functions of u and v in some domain, and that the Jacobian determinant

$$J = \frac{\partial u'}{\partial u}\frac{\partial v'}{\partial v} - \frac{\partial u'}{\partial v}\frac{\partial v'}{\partial v}$$

[1] M. A. Tissot, *Memoire sur la Representation des Surfaces* (Paris, Gautier—Villars, 1881). An application of Tissot's work in English can be found in: F. J. Marschner, "Structural Properties of Medium- and Small-scale maps," *Annals (Association of American Geographers)*, XXXIV, pp. 1–46.

does not vanish. These assumptions are not always obtained in photogrammetry. For example, all four sides of a building are usually not seen in an aerial photograph and thus the transformation contains a discontinuity at this point and is not one-to-one. Tissot's results will not hold in the vicinity of such discontinuities.

Tissot showed that for every non-conformal transformation there is one, and only one, set of orthogonal curves on the original which remains orthogonal on the image. Tissot went on to demonstrate that the maximum and minimum linear-scale change occurs in these two orthogonal directions. In addition, it is easily shown that there exists an affine relation between infinitesimally near points on the original and on the image. That is, the equations for the transformation at any point u_0, v_0 cam be written in the form

$$du' = \frac{\partial f_1}{\partial u_0} du - \frac{\partial f_1}{\partial v_0} dv$$

$$dv' = \frac{\partial f_2}{\partial u_0} du - \frac{\partial f_2}{\partial v_0} dv.$$

Since the partial derivatives at any point are constants this pair of equations can be rewritten as

$$du' = a_{11}du + a_{12}dv$$

$$dv' = a_{21}du + a_{22}dv,$$

which are the equations of an affine transformation.

The image of a circle under an affine transformation in general is an ellipse. As a consequence, an infinitesimally small circle on the original becomes an (infinitesimally small) ellipse on the image. This ellipse is now known as Tissot's Indicatrix. It can be shown that the axes of this ellipse are the lines along which the maximum and minimum amounts of change in scale are located. The semi-major and the semi-minor axes of this ellipse are designated by a and b, respectively; that is

$$\frac{ds'_{(\alpha)}}{ds} = a \text{ and } \frac{ds'_{(\alpha+\pi/2)}}{ds} = b.$$

Areal distortion (S), by definition, is the ratio of the element of area on the image to the element of area on the original, in other words, $S = dA'/dA$. Tissot's indicatrix yields an equivalent measure of the areal distortion as the product of the semi-axes, $S = ab$. For equal-area transformations $ab = 1$, and, since the maximum and minimum change of scale occurs in two orthogonal directions, there will be a pair of (non-orthogonal) curves along which the scale remains unchanged, the so-called isoperimetric lines. The amount by which angles between intersecting curves are altered is not given by the eccentricity of the ellipse, but the maximum amount by which angles are modified is given by

$$w = 2 \text{ arc sin } \frac{a - b}{a + b} \cdot$$

Perhaps the most practical result is that the local distortion can be completely specified by determination of the values for the two axes of Tissot's indicatrix. These are functions of the transformation and in general vary from point to point; i.e.,

$$a = g(u, v)$$

$$b = h(u, v).$$

There are certain special cases worth noting. If a transformation is conformal, every pair of orthogonal lines on the original is orthogonal on the image. On conformal transformations of the sphere the scale alteration varies from point to point but is the same in all directions about a point. In this case Tissot's indicatrix is a circle (of variable size since alteration of some geometric relations is unavoidable for map projections of a sphere or elliposid onto a plane).

On the other hand, it is possible to have transformations from one plane to another plane in which there is no alteration of geometric relations (the so-called Euclidean Group). Such a transformation is of necessity equal-area and conformal. This occurs only when a and b are unity and are independent of location ($a = b = 1$). This is the theoretical case against which camera lenses are usually evaluated. A lens with so-called radial distortion represents the special case for which one of the axes of Tissot's indicatrix lies along the radius vector.

All of Tissot's results hold for aerial photographs, as long as they represent one-to-one continuous images of the objects being photographed. This is true regardless of atmospheric refractions, lens distortion, displacement due to topography, tilt, warping of film, etc. The important distinction in practical photogrammetry of course is that it is usually not possible to calculate the values for Tissot's indicatrix on *a priori* grounds. Tissot's indicatrix sheds light on the problem of distortion, however, and may prove to be of practical importance in photogrammetry, just as it is important in the study of map projections.