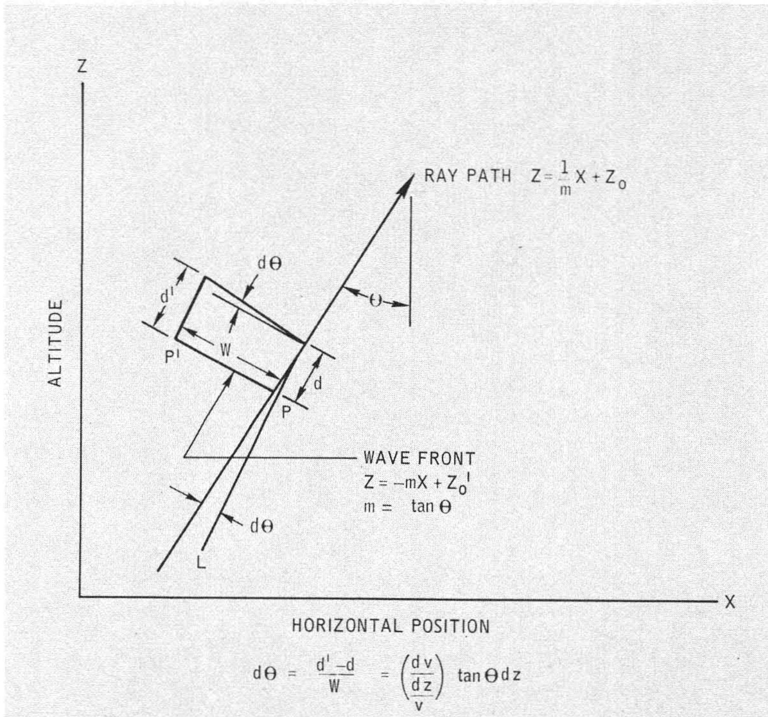


FIG. 1. Origin of atmospheric refraction.

DR. SIDNEY BERTRAM
The Bunker-Ramo Corp.
Canoga Park, Calif.

Atmospheric Refraction

INTRODUCTION

IT IS WELL KNOWN that the geometry of aerial photographs may be appreciably distorted by refraction in the atmosphere at the time of exposure, and that to obtain maximum mapping accuracy it is necessary to compensate this distortion as much as knowledge permits. This paper presents a solution of the refraction problem, including a hand calculation for the ARDC Model Atmosphere, 1959. The effect of the curvature of the earth on the refraction problem is analyzed separately and shown to be negligible, except for rays approaching the horizontal. The problem of determining the refraction for a practical situation is also discussed.

While atmospheric refraction has been discussed in the photogrammetric literature,[1,2,3] it is believed that the treatment presented herein represents a new and interesting approach to the problem.

THEORETICAL DISCUSSION

The origin of atmospheric refraction can be seen by reference to Figure 1. A light ray L is shown at an angle θ to the vertical in a medium in which the velocity of light v varies

[1] A. H. Faulds and Robert H. Brock, Jr., "Atmospheric Refraction and its Distortion of Aerial Photography," PHOTOGRAMMETRIC ENGINEERING Vol. XXX, No. 2, March 1964.

[2] H. H. Schmid, "A General Analytical Solution to the Problem of Photogrammetry," *Ballistic Research Laboratories Report No. 1065*, July 1959 (ASTIA 230349).

[3] D. C. Brown, "A Treatment of Analytical Photogrammetry," *RCA Data Reduction Technical Report No. 39*, AFMTC-TR-57-22, 20 August 1957 (ASTIA 124144).

with the altitude Z. The light ray is shown following a line $Z=(1/m)\ X+Z_0$, where $m=\tan\theta$; the wave front, perpendicular to the ray path, is then given by $Z=-mX+Z_0'$. At a distance W along the wave front the velocity is higher because of the increase in altitude; this results in a tipping of the wave front and, hence, a curvature of the ray path.

If $v=v(Z)$ expresses the variation of the velocity as a function of altitude, then

$$v(Z') = v(Z) + \left(\frac{dv}{dZ}\right) W \sin\theta \qquad (1)$$

where (dv/dZ) is the rate of change of velocity with altitude and $W \sin\theta$ is the vertical displacement between P (at altitude Z) and P' (at altitude Z'). The ray at P travels a distance d in time (d/v); in this same time the ray at P' travels a distance

$$d' = \left[v + \left(\frac{dv}{dZ}\right) W \sin\theta\right] \frac{d}{v} \qquad (2)$$

leading to a curvature of the path with a deviation angle $d\theta$ given by

$$d\theta = \frac{d' - d}{W} = \frac{\left(\frac{dv}{dZ}\right)}{v} d \sin\theta = \frac{\left(\frac{dv}{dZ}\right)}{v} \tan\theta\, dZ. \qquad (3)$$

If the element producing the refraction is at a height Z, the bending, $d\theta$, will produce a tangential displacement $(Z/\cos\theta)\ d\theta$ at the ground; for an observation point at a height Z_0 this will be seen as an angular deviation of $(Z/Z_0)d\theta$. The total refraction as viewed from Z_0 is given by the integral of $(Z/Z_0)d\theta$; thus

$$\delta\theta = \frac{1}{Z_0}\int_0^{Z_0} Z \frac{\left(\frac{dv}{dZ}\right)}{v} \tan\theta\, dZ. \qquad (4)$$

It is also useful to express this in terms of the horizontal position X from the object point on the ground. This is given by

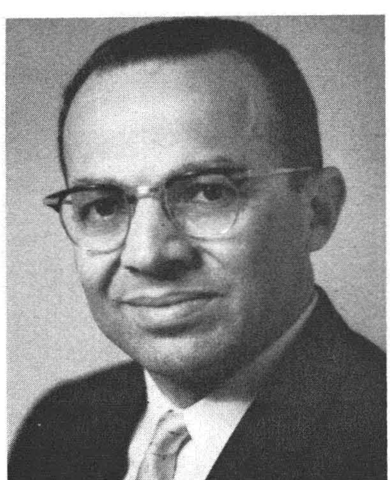
$$\delta\theta = \frac{1}{X_0}\int_0^{X_0} X \frac{\left(\frac{dv}{dX}\right)}{v} \tan\theta\, dX. \qquad (5)$$

Equation 5 is used in the analysis of the effects of the curvature of the earth on the refraction, because X is more nearly independent of the curvature than is Z.

Equation 4 may be considerably simplified by observing that both v and $\tan\theta$ are essentially constant and, hence, may be taken outside the integral (v varies less than 0.1% while $\delta\theta$ is less than $10^{-4}\tan\theta$ radians). The resulting equation is

$$\delta\theta = \frac{\tan\theta}{vZ_0}\int_0^{Z_0} Z \left(\frac{dv}{dZ}\right) dZ. \qquad (6)$$

ABSTRACT: *The distortion of aerial photographs caused by atmospheric refraction is shown to be readily calculable in terms of an integral involving the variation with altitude of the velocity of light. The integral is evaluated numerically for the ARDC Model Atmosphere, 1959, using an accepted relationship between velocity and atmospheric density. The effect of local atmospheric conditions is explored by calculating the refraction for the extreme cases of an arctic winter and for the tropics. These are shown to yield refractions of comparable magnitude to that found for the ARDC model. It is concluded that adequate corrections for the distortion can usually be made using the data given. However, more precise corrections can be made using the technique described whenever it is practical to determine the variation of air density as a function of altitude.*


DR. SIDNEY BERTRAM

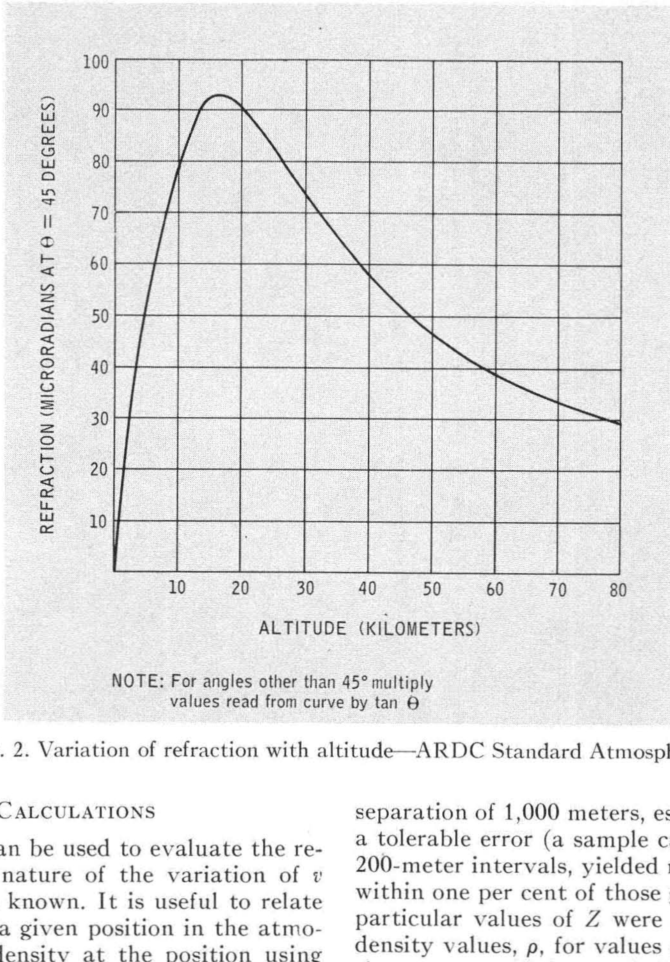


FIG. 2. Variation of refraction with altitude—ARDC Standard Atmosphere.

CALCULATIONS

Equation 6 can be used to evaluate the refraction if the nature of the variation of v with altitude is known. It is useful to relate the velocity at a given position in the atmosphere to the density at the position using the formula[4] $\mu = 1 + 0.000226\rho$ so that

$$\frac{v}{c} = \frac{1}{\mu} \approx 1 - 0.000226\rho \qquad (7)$$

where μ is the index of refraction at a point where the density is ρ in kg./m.^{3} and c is the velocity of light in free space; here $v/c \approx 1$, but $dv/dZ = -0.000226\,(d\rho/dZ)c$. Values for ρ as a function of altitude have been tabulated for the ARDC Standard Atmosphere, 1959[5] and have been used with Equation 6 to calculate the refraction characteristic shown in Figure 2. Since the procedure may be used whenever the density as a function of altitude is known, it is shown in the detailed calculations of Tables 1 and 2; an explanation of the tables follows.

The various entries in the table are at a Z separation of 1,000 meters, estimated to yield a tolerable error (a sample calculation, using 200-meter intervals, yielded refraction values within one per cent of those given here). The particular values of Z were selected so that density values, ρ, for values on either side of the entry could be read directly from the ARDC tables. Thus, for the first entry at $Z = 500$, density values were obtained for $Z = 0(\rho = 1.225)$ and $Z = 1{,}000(\rho = 1.112)$. The difference between these values is taken to be the rate of change of density at $Z = 500$, i.e., $\rho' = (1.225 - 1.112) = 0.113$ in kg./m.^{3}/1,000 meters. As shown by Equation 7, this is multiplied by 0.000226 to obtain the velocity change for the 1,000-meter interval; this multiplication is performed in the last column.

The integration is carried out using the trapezoidal rule. Thus, for a given interval, the contribution to the integral is

$$\frac{(\rho'Z)_n + (\rho'Z)_{n+1}}{2}\,\frac{dZ}{\Delta Z} \qquad (8)$$

where $(\rho'Z)_n$ is the product of ρ' (as determined at Z_n) and Z_n, and $(\rho'Z)_{n+1}$ is the corresponding product for the next point. The two differentials dZ and ΔZ, have both been made 1,000 meters so $dZ/\Delta Z = 1$; the third column is, therefore, $\rho'Z/2$. The fourth

[4] D. C. Brown, *op. cit.*, p. 36.

[5] R. A. Mizner, KSW Champion, and H. L. Pond, "The ARDC Model Atmosphere, 1959," *Air Force Surveys in Geophysics No. 115* (ASTIA 229482).

TABLE 1
REFRACTION FOR ARDC MODEL ATMOSPHERE
(1000-meter calculation interval)

Z	ρ	ρ'	$\rho' Z/2$	Σ	$\frac{1}{Z}\Sigma$	$\frac{226}{Z}\Sigma$ (*microradians*)
	1.225					
500		0.113	28	14	0.028	6.4
	1.112					
1,500		0.105	79	121	0.081	18.3
	1.007					
2,500		0.098	121	321	0.129	29.1
	0.909					
3,500		0.090	157	599	0.172	38.8
	0.819					
4,500		0.083	186	942	0.210	47.5
	0.736					
5,500		0.076	210	1,338	0.244	55.1
	0.660					
6,500		0.070	228	1,776	0.274	61.9
	0.590					
7,500		0.064	241	2,245	0.300	67.8
	0.526					
8,500		0.059	249	2,735	0.322	72.8
	0.467					
9,500		0.054	255	3,239	0.341	77.1
	0.413					
10,500		0.049	256	3,750	0.357	80.7
	0.365					
11,500		0.053	304	4,310	0.375	84.8
	0.312					
12,500		0.045	283	4,897	0.392	88.6
	0.267					
13,500		0.039	262	5,442	0.403	91.1
	0.228					
14,500		0.033	240	5,944	0.410	92.7
	0.195					
15,500		0.028	218	6,402	0.413	93.3
	0.166					
16,500		0.024	200	6,820	0.413	93.3
	0.142					
17,500		0.021	181	7,201	0.412	93.1
	0.122					
18,500		0.018	163	7,545	0.408	92.2
	0.104					
19,500		0.015	147	7,855	0.403	91.1
	0.089					
20,500		0.013	132	8,137	0.397	89.7
	0.076					

column is the current sum. Thus, for $Z = 1,500$ meters one has the sum to 500 meters (14), to which the contribution for the interval 500 to 1,000 has to be added (28+79); therefore, the value at 1,000 meters is (14+28+79)=121. The value at 500 meters is exceptional, first because the value at zero is zero, and second because the interval is only 500 meters so the contribution for the interval is taken at only half value ($\Sigma = 28/2 = 14$).

The next column has the sum divided by the current altitude; this is multiplied by the constant 226 in the last column to obtain the refraction in microradians (since the multiplier would be 0.000226 to obtain the angle in radians).

Table 1 includes up to 20,500 meters. The

TABLE 2
REFRACTION FOR ARDC MODEL ATMOSPHERE
(3,000-meter calculation interval)

Z	ρ	ρ'	$\rho' Z/2$	Σ	$\frac{1}{Z}\Sigma$	$\frac{226}{Z}\Sigma$ (*microradians*)
	0.1040					
20,500		0.0390	400	8,137	0.397	89.7
	0.0650					
23,500		0.0244	287	8,824	0.375	84.7
	0.0406					
26,500		0.0159	211	9,320	0.352	79.6
	0.0247					
29,500		0.0095	140	9,670	0.328	74.1
	0.0152					
32,500		0.0056	91	9,900	0.305	68.9
	0.0096					
35,500		0.0035	62	10,050	0.283	64.0
	0.0061					
38,500		0.0021	40	10,150	0.264	59.7
	0.00400					
41,500		0.00136	28	10,220	0.246	55.6
	0.00264					
44,500		0.00087	19	10,270	0.231	52.2
	0.00177					
47,500		0.00055	13	10,310	0.217	49.0
	0.00122					
50,500		0.00037	9	10,330	0.205	46.3
	0.00085					
53,400		0.00024	6	10,340	0.193	43.6
	0.00061					
56,500		0.00020	6	10,350	0.183	41.3
	0.00041					
59,500		0.00010	3	10,360	0.174	39.3
	0.00031					
62,500		0.00009	3	10,370	0.166	37.5
	0.00022					
65,500		0.00007	2	10,370	0.158	35.7
	0.00015					
68,500		0.00005	2	10,380	0.152	34.4
	0.00010					
71,500		0.00004	1	10,380	0.145	32.8
	0.00006					
74,500		0.00002	1	10,380	0.139	31.4
	0.00004					
77,500		0.00002	1	10,380	0.134	30.3
	0.00002					
80,500		0.00000	0	10,380	0.129	29.1
	0.00002					

calculations are extended to 80,500 meters in Table 2 using a calculation interval of 3,000 meters. Table 2 begins with the sum 8,137 found for 20,500 meters in Table 1. The ρ values used are for the altitude entry $\pm 1,500$ meters; thus, for the first entry ρ (19,000) $= 0.1040$ and ρ (22,000) $= 0.0650$ to yield $\rho' = (0.1040 - 0.0650) = 0.0390$. The other entries follow in similar manner to that described for Table 1.

The refraction angle as a function of altitude, as shown in the last columns of Tables 1 and 2, is plotted in Figure 2.

It is frequently necessary to adjust the refraction data to compensate for object points that have altitudes significantly different from sea level. The previous calculations could obviously have been started at any altitude to give the required result; fortunately it is possible to bypass this operation by a simple expedient. This procedure is first described by an example and then generalized.

Suppose it is desired to obtain the refraction for an object point at 1,500 meters where the camera station is at 10,500 meters. From Table 1 the refraction for an object point at sea level is 80.7 microradians, while the refraction of a sea level point as viewed from 1,500 meters would be 18.3 microradians. The 80.7 figure includes the 18.3 value, but scaled down by the ratio (1,500/10,500) because of the difference in viewpoint; therefore, the required value is

$$80.7 - 18.3\frac{1{,}500}{10{,}500} = 80.7 - 2.6 = 78.1 \text{ microradians}$$

The generalization of the above result follows directly:

$$\delta\theta = \delta\theta_c - \frac{Z_p}{Z_c}\delta\theta_p \quad (9)$$

where $\delta\theta_c$ is the refraction to sea level from the camera at elevation Z_c and $\delta\theta_p$ is the refraction to sea level from the object point at elevation Z_p.

As no appreciable refraction occurs at altitudes beyond those shown, the effective refraction at any higher altitude may be obtained by the same technique. This yields

$$\theta_c = (29.1 \times 80.5)/Z = 2.340/Z$$

were Z is in kilometers.

Corrections to the Calculations

It is useful to consider other factors that affect the refraction and, hence, might cause a deviation from the values calculated for the standard atmosphere, flat-earth calculation made above. There are good *a priori* reasons for thinking that the effects of the curvature of the earth are negligible in this context since the effects are only appreciable at a great distance and the refraction is multiplied by the ratio Z_p/Z_c as above. This is confirmed in the analysis of the next section.

CURVED EARTH EFFECTS

The effects of the finite radius of curvature of the earth on the refraction may be found starting with Equation 5; the curved earth geometry is expressed by

$$Z = \frac{X}{m} + \frac{(X_0 - X)^2}{2R}$$

and

$$\theta_v = \theta + \frac{(X_0 - X)}{R}$$

where X is the distance to the object point, θ is the angle of the light ray with respect to the vertical at the camera, and θ_v the angle at a point distant $(X_0 - X)$ from the camera; R is the radius of the earth. Then

$$\frac{dv}{dX} = \left(\frac{dZ}{dX}\right)\left(\frac{dv}{dZ}\right) = \left[\frac{1}{m} - \frac{(X_0 - X)}{R}\right]\frac{dv}{dZ}$$

and, using the first two terms of a Taylor expansion,

$$\tan\left(\theta + \frac{X_0 - X}{R}\right) \approx \tan\theta + \frac{(X_0 - X)}{R}\sec^2\theta.$$

Substituted in Equation 5, these yield

$$\delta\theta \approx \frac{1}{vX_0}\int_0^{X_0} X\left[\operatorname{ctn}\theta - \frac{(X_0 - X)}{R}\right]\left(\frac{dv}{dZ}\right) \cdot \left[\tan\theta + \frac{(X_0 - X)}{R}\sec^2\theta\right] dX.$$

Expansion of the above and neglecting the term in $(X_0 - X)^2$ next yields

$$\delta\theta \approx \frac{1}{vX_0}\int_0^{X_0} X\left(\frac{dv}{dZ}\right) dX + \frac{\operatorname{ctn}\theta}{vRX_0}\int_0^{X_0}(X - X_0)X\left(\frac{dv}{dZ}\right) dX.$$

If Z is again set as the independent variable this becomes, finally,

$$\delta\theta \approx \frac{\tan\theta}{vZ_0}\int_0^{Z_0} Z\left(\frac{dv}{dZ}\right) dZ + \frac{\tan\theta}{vZ_0}\int_0^{Z_0}\frac{(Z - Z_0)Z}{R}\left(\frac{dv}{dZ}\right) dZ.$$

The first term, independent of R, is the solution for a plane earth. The integrand of the second term is everywhere less than that of the first by a factor $(Z - Z_0)/R$ and the integral is reduced by at least this amount. For an elevation of 64 kilometers, for example $(Z - Z_0)/R \approx 0.01$; yielding a correction of less than 1% over the plane earth value. It is concluded, therefore, that it is valid to neglect the curvature of the earth in calculating the refraction for ordinary photogrammetric situations. For situations where θ is so large that the second term becomes an appreciable part of the first, the second integral could be evaluated numerically to obtain the desired correction.

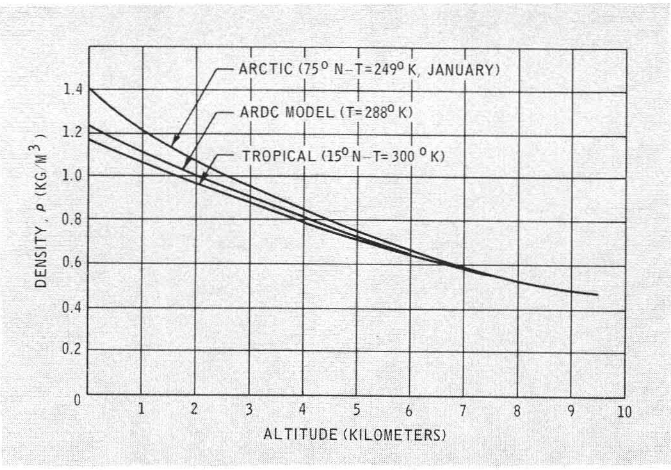


FIG. 3. Variation of atmospheric density with altitude for various atmospheres.

Variation of Refraction with Region and Season

It is possible to place some bounds on the refraction for normal atmospheric conditions at various places on the earth and various seasons of the year using data given by Cole and Cantor.[6] The density function for two extreme situations, as given in the reference, are plotted in Figure 3 along with the density function for the ARDC Model Atmosphere. Table 3 shows the calculation for refraction for the arctic in January, while Table 4 shows the calculations for refraction for a tropical location. The resulting refraction functions are shown in Figure 4; values for altitudes greater than 9,000 meters were calculated from the data for the standard atmosphere.

[6] Cole and Cantor, "Air Force Supplemental Atmosphere to 90 km.," December 1963.

TABLE 3

REFRACTION, ARCTIC (75°N) JANUARY

Z	ρ	ρ'	$\rho' Z/2$	Σ	$\frac{1}{Z}\Sigma$	$\frac{226}{Z}\Sigma$ (*microradians*)
0						
	1.300					
1,000		0.160	80	80	0.080	18
	1.140					
2,000		0.127	127	287	0.143	32
	1.013					
3,000		0.111	167	581	0.194	44
	0.902					
4,000		0.104	208	956	0.238	54
	0.798					
5,000		0.093	233	1,397	0.279	63
	0.705					
6,000		0.080	240	1,870	0.312	71
	0.625					
7,000		0.072	252	2,362	0.338	76
	0.553					
8,000		0.061	244	2,858	0.357	81
	0.492					
9,000		0.054	243	3,345	0.372	84
	0.438					
10,000						

TABLE 4

REFRACTION, TROPICAL (15°N)

Z	ρ	ρ'	$\rho' Z/2$	Σ	$\frac{1}{Z}\Sigma$	$\frac{226}{Z}\Sigma$ (*microradians*)
0						
	1.113					
1,000		0.097	48	48	0.048	11
	1.016					
2,000		0.093	93	189	0.094	21
	0.923					
3,000		0.089	133	415	0.138	31
	0.834					
4,000		0.078	156	704	0.176	40
	0.756					
5,000		0.073	183	1,043	0.209	47
	0.683					
6,000		0.070	210	1,436	0.239	54
	0.613					
7,000		0.061	214	1,860	0.266	60
	0.552					
8,000		0.061	244	2,318	0.289	65
	0.491					
9,000		0.053	239	2,801	0.311	70
	0.438					
10,000						

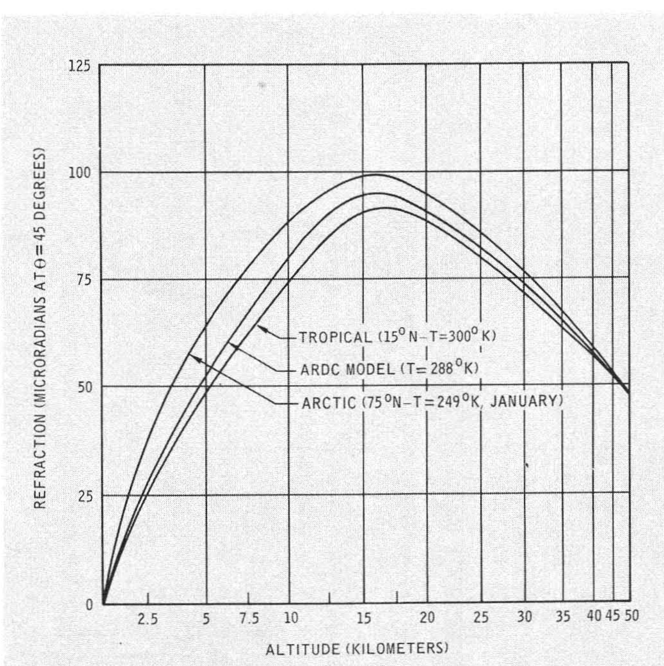


FIG. 4. Variation of refraction with altitude for different atmospheres.

It will be noted that the arctic air increases the surface density, resulting in a greater rate of change of density near the surface (it drops rapidly to meet the upper air density that is essentially independent of position or season). The refraction is significantly higher under these circumstances; the difference between the refraction in the tropics and that for the standard atmosphere is much smaller.

The significance of atmospheric refraction for a given operation depends on the nature of the operation. For a camera at 10,000 feet (3,000 meters) the refraction would be about 33 microradians at $\theta=45$ degrees; if the camera was vertically oriented, with a 6-inch lens (150,000-micron) this would indicate a displacement on the film of

$$33 \times 10^{-6} \times (150{,}000\sqrt{2}) \times \sqrt{2} \approx 10 \text{ microns}$$

Since this is large compared to the accuracy of comparator measurements, it should be corrected for best results.

Summary and Conclusion

It has been shown that the effects of atmospheric refraction, as seen from an aerial camera, can be calculated in a simple manner if the density as a function of altitude is known. The method has been used to calculate the refraction characteristics of the ARDC Model Atmosphere, 1959, and of two extreme atmospheres, one for the tropics and a second for an arctic winter situation. These calculations permit a number of useful generalizations.

Inasmuch as the sea level pressure is essentially uniform over the earth, the corresponding density is a function of the surface temperature in accordance with the gas laws; more precisely, the density is inversely proportional to the absolute temperature, and a knowledge of the surface temperature suffices to determine the surface density. If a normal temperature distribution as a function of altitude is assumed, this one point will establish the corresponding density function with relationship to those drawn in Figure 3. It is expected that the resulting curve will be between the extremes drawn. A calculation of the refraction characteristic should then yield a curve lying between the extreme curves of Figure 4. As the refraction ordinarily causes only a very small distortion, it is likely that the "nominal" refraction curve for the ARDC Model Atmosphere will yield values of sufficient accuracy for most purposes.

While quantitative data describing temperature inversions have not been found, it is possible to discuss such situations in a qualitative manner. Temperature inversions are characterized by an increasing temperature with increasing altitude at low altitudes, in contrast with the decreasing temperature found in a normal atmosphere. In accordance with the gas laws, this situation must be accompanied by a lower than usual air density with increasing altitude; hence, the density must decrease faster than normal near the surface but flatten out to meet the standard curve by, say, 7,500 meters. The result is a larger than normal refraction near the surface, but once above the inversion layer the refraction curve will have a lower slope and the maximum value of refraction will be lower.

It should be possible to use the nominal refraction curves for most applications. However, if better accuracy is desired, a modest amount of additional data, beginning with the surface temperature and augmented, where possible, by temperature samples in the region below 5,000 meters, should permit a significant improvement in accuracy for a given operation.