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Determination of Quadric Surfaces from Two Projective Views

Parameters which define a quadric surface in three-space may be found from measurements of image coordinates of the surface's limb.

ABSTRACT: *A projective view of a quadric defines a limb which is a curve on the surface. The projection of this limb is observable on the view, and coordinates of points on this projection may be measured. The problem is to determine defining parameters of the surface from such measurements gathered from two views and the camera model of each view.*

The given method of solution is to write the equation of the limb image as $\mathbf{u}^T \mathbf{M}\, \mathbf{u} = 0$ *where* $\mathbf{u}$ *is a* 3×1 *vector whose first two components are measured image coordinates and whose third component is 1, and* $\mathbf{M}$ *is a* 3×3 *matrix which is dependent upon the unknown parameters of the quadric and the known camera model. Thus, each measured* $\mathbf{u}$ *furnishes an equation in the unknown parameters of the surface. Sufficient measurements from two views then imply the parameters. A method of determining the parameters of a plane which truncates a quadric is also given.*

INTRODUCTION

AN IMPORTANT APPLICATION of photogrammetry is that of ascertaining the significant dimensions of a cultural scene from two or more close range images of the scene. If the scene is composed of portions of planar surfaces, as many cultural scenes are, then the significant dimensions of the scene may be inferred from the image coordinates of the scene's vertices. Of course, all cultural scenes are not entirely composed of planar surface segments. Many such scenes contain portions of quadric surfaces, that is, surfaces that are or may be approximated as a polynomial of second degree in three variables. In practice, such surfaces are usually spheres, right circular cylinders, or right circular cones. However, methodology of obtaining parameters which define these surfaces is easily generalized to all quadrics.

A quadric surface will image on a projective view as a simply connected region. This region will have a boundary curve which will be called here the limb image while the curve on the quadric which is the pre-image of the limb image is called the limb. Two views of the surface will yield two limb images. Image coordinates of points on these curves can be measured. The problem is to determine the parameters which define the surface from these measurements and the camera models which define each projective view.

Figure 1 shows two images of an ellipsoid. In this case, the image of each limb is an ellipse, and fixing these ellipses and the two camera models does define the ellipsoid in three-space. However, the problem of determining parameters which define the ellipsoid from measured coordinates on the two limb images is more complicated then that of defining planar surfaces from image measurements. This occurs because, in general, a point on the limb image in one view does not have a match point on the limb image in the other view. Thus, use of matched points cannot be made and the problem must be attacked more indirectly.

The form of the camera model assumed here is the most general form which images every straight line, not lying in a singular plane, in three-space onto a straight line in the image. This form may be written as (Kober and Grosch, 1980)

$$u = \frac{\mathbf{A}_1 \mathbf{X} + a_1}{\mathbf{A}_3 \mathbf{X} + a_3}$$

$$v = \frac{\mathbf{A}_2 \mathbf{X} + a_2}{\mathbf{A}_3 \mathbf{X} + a_3} \qquad (1)$$

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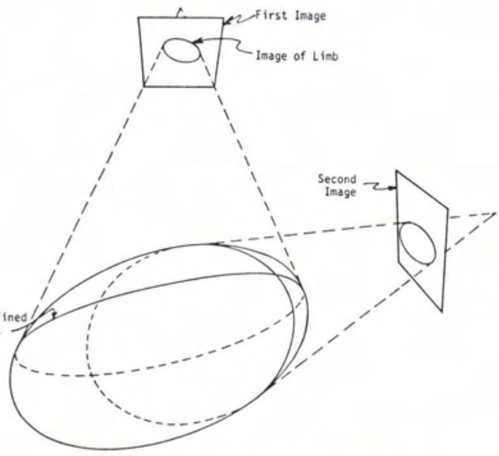

FIG. 1. Two views of an ellipsoid.

where $\mathbf{X}$ is a vector of three-space coordinates, (u, v) is the image of $\mathbf{X}$, and

$\mathbf{A}$ is a 3×3 non-singular matrix;
a_1, a_2, a_3 are scalars; and
$\mathbf{A}_i$ is the ith row of $\mathbf{A}$.

If the image plane is a subspace of three-space, and if the origin of the image coordinate system is at a preferred point (usually called the principal point), then the model (Equation 1) may be written as

$$u = -F \frac{\mathbf{A}_1 \mathbf{X} + a_1}{\mathbf{A}_3 \mathbf{X} + a_3}$$
$$v = -F \frac{\mathbf{A}_2 \mathbf{X} + a_2}{\mathbf{A}_3 \mathbf{X} + a_3} \qquad (2)$$

where $\mathbf{A}$ in Equation 2 is a rotation matrix (i.e., $\mathbf{A}^T = \mathbf{A}^{-1}$) and F is the focal length. Thus, the Equation 2 model is a special case of the Equation 1 model. The more general form (Equation 1) will be assumed here. The matrix $\mathbf{A}$ and the scalars a_1, a_2, a_3 are assumed known.

General Method

An outline of the general method will first be given, the details will then be developed.

An equation for a general quadric may be written in terms of an unknown parameter vector, $\mathbf{p}$. For the more general case, $\mathbf{p}$ has nine components but for special cases $\mathbf{p}$ may have fewer components. For example, if it were known that the surface is a sphere, then $\mathbf{p}$ has only four components. In any event, given camera model (1) it is then possible to write the equation for the limb image of the surface in the form

$$f(u,v,\mathbf{p}) = 0 \qquad (3)$$

where (u,v) are the image coordinates of a point on the limb image.

Once this equation has been written, each measured pair, (u_i, v_i), of limb image coordinates yields one equation in $\mathbf{p}$. Nine such measured pairs yield nine equations in nine unknowns. However, the solution of these equations is not unique; thus, $\mathbf{p}$ cannot be determined from one image alone. As expected, a second image of the surface must be utilized to determine $\mathbf{p}$.

Limb Image

The first step, that of writing the equation of the limb image of a general quadric surface, will now be developed; that is, Equation 3 will be explicitly written. It will be shown that this equation is of the form $\mathbf{u}^T \mathbf{M} \mathbf{u} = 0$, where $\mathbf{u}^T = (u,v,1)$ and $\mathbf{M}$ is a 3×3 matrix which depends on the camera model and the quadric. Apparently, the explicit form of this equation is new here.

A general quadric surface may be written as

$$\mathbf{X}^T \mathbf{B} \mathbf{X} + 2\mathbf{b}^T \mathbf{X} = c_1$$

where

$\mathbf{B}$ is a 3×3 symmetric non-zero matrix,
$\mathbf{b}$ is a 3×1 vector, $\mathbf{b}^T$ is the transpose of $\mathbf{b}$, and
c_1 is a scalar.

Two cases exist (Jeger and Echmann, 1967). If the system of equations

$$\mathbf{B}\,\mathbf{h} + \mathbf{b} = 0 \qquad (4)$$

has a solution for $\mathbf{h}$, then the quadric may be written as

$$(\mathbf{X} - \mathbf{h})^T \mathbf{B}(\mathbf{X} - \mathbf{h}) = c_1 - \mathbf{b}^T\mathbf{h} = c. \qquad (5)$$

However, if Equation 4 has no solution, then the surface is a paraboloid and

$$\text{rank } \mathbf{B} < 3, \text{ rank } \mathbf{B} \neq \text{rank } (\mathbf{B},\mathbf{b}).$$

First, assume the surface is not parabolic so it can be written in the form of Equation 5.

If the camera model (Equation 1) is assumed, given any (u,v), the pre-image of these coordinates (i.e., all points in three-space which image at (u,v)) is

$$\mathbf{X} = \mathbf{A}^{-1}(\lambda\,\mathbf{u} - \mathbf{a})$$

where

$$\mathbf{u} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{ and}$$

λ is arbitrary.

If $\mathbf{X}$ lies on the surface, it must satisfy Equation 5, thus,

$$(\lambda\mathbf{u} - \mathbf{a} - \mathbf{k})^T \mathbf{A}^{-T} \mathbf{B}\mathbf{A}^{-1} (\lambda\mathbf{u} - \mathbf{a} - \mathbf{k}) = c \qquad (6)$$

where $\mathbf{k} = \mathbf{A}\mathbf{h}$ and $-T$ represents the inverse of the transpose. Now, for any given $\mathbf{u}$, Equation 6 has two, one, or zero solutions for λ. However, the limb image of the quadric are those $\mathbf{u}$ which yield precisely one value of λ. Therefore, $\mathbf{u}$ must satisfy

$$\mathbf{u}^T (\mathbf{K}\mathbf{d}\mathbf{d}^T\mathbf{K} - \mathbf{d}^T\mathbf{K}\mathbf{d}\,\mathbf{K} + c\mathbf{K})\mathbf{u} = 0 \qquad (7)$$

where

$$\mathbf{K} = \mathbf{A}^{-T}\mathbf{B}\mathbf{A}^{-1} \qquad (8)$$
$$\mathbf{d} = \mathbf{a} + \mathbf{k} = \mathbf{a} + \mathbf{A}\mathbf{h}. \qquad (9)$$

Equation 7 is the equation of the limb image of the surface given by Equation 5; however, this equation can be simplified so that it becomes

$$\mathbf{u}^T\,\mathbf{M}\,\mathbf{u} = 0 \qquad (10)$$

where

$$\mathbf{M} = \mathbf{Q}\tilde{\mathbf{B}}\,\mathbf{Q}^T - \mathbf{c}\,\tilde{\mathbf{A}}\,\mathbf{B}\,\tilde{\mathbf{A}}^T.$$

Here

$\tilde{\mathbf{B}} =$ cof $\mathbf{B}$ = matrix composed of the cofactors of $\mathbf{B}$,
$\tilde{\mathbf{A}} =$ cof $\mathbf{A}$,

$$\mathbf{Q} = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} \mathbf{A}, \text{ and}$$

$$(11)$$

$$\mathbf{d} = \quad \mathbf{a} + \mathbf{A}\,\mathbf{h} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Equation 10 is the equation of the limb image of all non-parabolic quadrics. It is interesting that its derivation was obtained under the assumption that $\mathbf{A}$ was non-singular. However, it also holds if $\mathbf{A}$ is singular (so it can also be used for parallel projection camera models).

Before extending the development to the parabolic case, it is instructive to write Equation 10 for three special quadrics, *viz.*, sphere, right circular cylinder, and right circular cone. These are the quadrics most likely to be encountered in a cultural scene.

Three Special Cases

If the quadric is a sphere, $\mathbf{B}$ and c of Equation 5 become

$$\mathbf{B} = \mathbf{I}\ (3 \times 3 \text{ identity})$$
$$c = r^2$$

where r is the sphere's radius. So, Equation 10 becomes

$$\mathbf{u}^T\,(\mathbf{Q}\,\mathbf{Q}^T - r^2\,\tilde{\mathbf{A}}\tilde{\mathbf{A}}^T)\,\mathbf{u} = 0$$

where $\mathbf{Q}$ is given by Equation 11, and $\mathbf{u}^T = (u, v, 1)$. Thus, the limb image of a sphere is an ellipse whose center, in general, is not the image of the sphere's center.

A general right circular cylinder has an equation which may be written as

$$(\mathbf{X} - \mathbf{h})^T\,\mathbf{R}^T\,\mathbf{D}\,\mathbf{R}\,(\mathbf{X} - \mathbf{h}) = r^2$$

where $\mathbf{R}^T\mathbf{R} = \mathbf{I}$,

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and}$$

$$\mathbf{R}_3\,\mathbf{h} = 0.$$

The direction of the cylinder's axis is $\mathbf{R}_3$ (the third row of $\mathbf{R}$) as shown in Figure 2.

Thus, for this case, Equation 10 becomes

$$\mathbf{u}^T\,\mathbf{M}\,\mathbf{u} = 0$$

where

$$\mathbf{M} = \mathbf{Q}\mathbf{R}^T{}_3\,\mathbf{R}_3\,\mathbf{Q}^T - r^2\,\tilde{\mathbf{A}}\,\mathbf{R}^T\,\mathbf{D}\,\mathbf{R}\,\tilde{\mathbf{A}}^T. \qquad (12)$$

But, intuition and experience tells us that the limb image of a right circular cylinder is a pair of straight lines, as shown in Figure 2. This is indeed the case for *any* cylinder or cone.

In general, the graph of $\mathbf{u}^T\,\mathbf{M}\,\mathbf{u} = 0$ is a pair of straight lines if there exists a $\mathbf{q} \neq 0$ such that

$$\mathbf{M}\,\mathbf{q} = 0 \text{ (zero vector)}.$$

In this case, the equation of the straight lines is given by

$$M_{11}\,(q_3 u - q_1) + (M_{12} \pm \sqrt{M_{12}^2 - M_{11}M_{22}})\,(q_3 v - q_2) = 0$$

if $q_3 \neq 0$, if $q_3 = 0$

$$M_{11}\,u + M_{12}\,v = -M_{13} \pm \sqrt{M_{13}^2 - M_{11}\,M_{33}}.$$

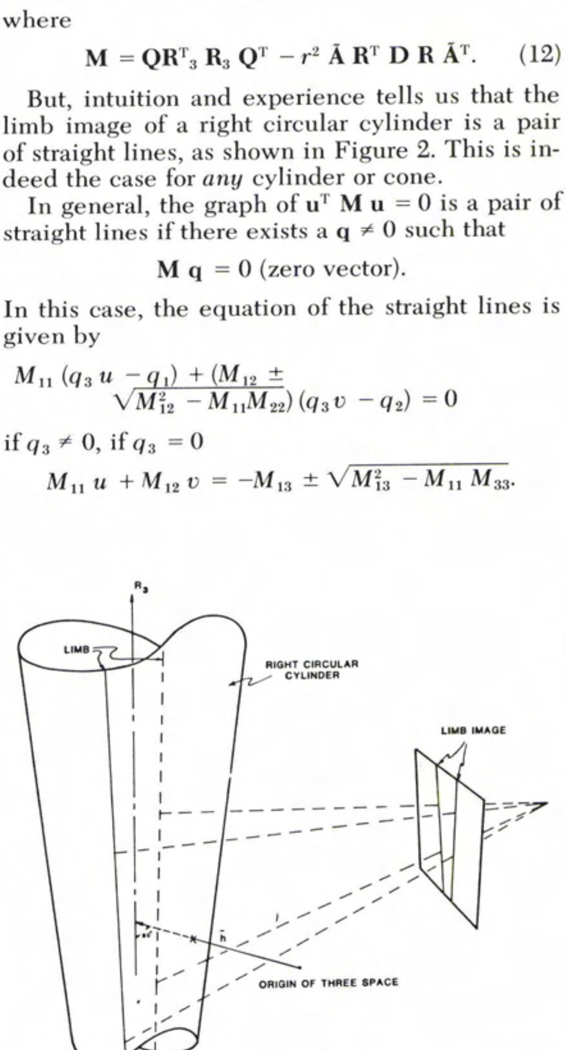

Fig. 2. Right circular cylinder and its limb image.

For the case in which $\mathbf{M}$ is given by Equation 12, $\mathbf{q}$ does exist and is given by

$$\mathbf{q} = \mathbf{A}\ \mathbf{R}_3^{\mathrm{T}}.$$

As a third special case, consider a right circular cone. A general right circular cone may be written

$$(\mathbf{X} - \mathbf{h})^{\mathrm{T}}\ \mathbf{R}^{\mathrm{T}}\mathbf{D}\ \mathbf{R}(\mathbf{X} - \mathbf{h}) = 0$$

where

$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, θ is the half-vertex angle,

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\tan^2\theta \end{pmatrix},$$

and $\mathbf{R}_3$ is the direction of the cone's axis.

So, in this case

$$\mathbf{M} = \mathbf{Q}\mathbf{R}^{\mathrm{T}} \begin{pmatrix} \tan^2\theta & 0 & 0 \\ 0 & \tan^2\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathbf{R}\mathbf{Q}^{\mathrm{T}}.$$

For this case,

$$\mathbf{M}\ \mathbf{d} = 0$$

where $\mathbf{d}$ is given by Equation 7. So, the limb image of a right-circular cone is also a pair of straight lines. In general, the limb images of all cylinders and cones are straight line pairs.

Parabolic Quadrics

If the quadric is parabolic, it cannot be written in the form of Equation 5; hence, the equation for its limb image is different than from Equation 10, which applies only to non-parabolic quadrics. The general equation of the limb image of parabolic quadrics will now be considered.

The equation of a general parabolic may be written as

$$(\mathbf{X} - \mathbf{h})^{\mathrm{T}}\ \mathbf{R}^{\mathrm{T}}\ \mathbf{D}\ \mathbf{R}\ (\mathbf{X} - \mathbf{h}) = 2\mathbf{R}_3\ (\mathbf{X} - \mathbf{h})$$

where

$\mathbf{R}$ is a rotation matrix ($\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$),

$\mathbf{R}_3$ is the third row of $\mathbf{R}$ and also a unit vector in the direction of the paraboloid's axis, and

$$\mathbf{D} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, these surfaces are specified by eight parameters instead of nine as is required for a general non-parabolic quadric.

The development of the equation of the limb image proceeds as before with the result

$$\mathbf{M} = \alpha_1\ \alpha_2\ \mathbf{Q}\ \mathbf{R}_3^{\mathrm{T}}\ \mathbf{R}_3\ \mathbf{Q}^{\mathrm{T}} - \mathbf{A}\ \mathbf{R}_3^{\mathrm{T}}\ \mathbf{R}_3\ \tilde{\mathbf{A}}^{\mathrm{T}} - 2\ \mathbf{Q}\ (\alpha_1\ \mathbf{R}_2^{\mathrm{T}}, -\alpha_2\ \mathbf{R}_1^{\mathrm{T}}, 0)\ \mathbf{R}\ \tilde{\mathbf{A}}^{\mathrm{T}} \qquad (13)$$

where, as before, $\mathbf{Q}$ is given by Equation 11. Note that the matrix $\mathbf{M}$ is now not symmetric. However, if desired, the symmetric part of this matrix may be used because a quadric form associated with a skew-symmetric matrix is identically zero.

If the surface is a paraboloid of revolution, $\alpha_1 = \alpha_2 = \alpha$ so that Equation 13 becomes

$$\mathbf{M} = \alpha^2\mathbf{Q}_3\ \mathbf{R}_3^{\mathrm{T}}\ \mathbf{R}\ \mathbf{Q}^{\mathrm{T}} - 2\ \alpha\ \mathbf{Q} \begin{pmatrix} 0 & -R_{33} & R_{32} \\ R_{33} & 0 & R_{31} \\ -R_{32} & R_{31} & 0 \end{pmatrix} \tilde{\mathbf{A}}^{\mathrm{T}} - \tilde{\mathbf{A}}\ \ \mathbf{R}_3^{\mathrm{T}}\ \mathbf{R}_3\ \tilde{\mathbf{A}}^{\mathrm{T}}$$

However, if the quadric is a cylincrical paraboloid, then α_1 or α_2 is zero. Suppose $\alpha_2 = 0$, then

$$\mathbf{M} = (2\ \alpha_1\ \mathbf{Q}\ \mathbf{R}_2^{\mathrm{T}}\ \mathbf{R}_1 + \tilde{\mathbf{A}}\ \mathbf{R}_3^{\mathrm{T}}\ \mathbf{R}_3)\ \tilde{\mathbf{A}}^{\mathrm{T}}$$

where $\mathbf{R}_2$ is the direction of the cylinder's elements.

Because

$$\mathbf{M}\ \mathbf{A}\ \mathbf{R}_2^{\mathrm{T}} = 0,$$

the limb image is two straight lines.

Procedure

Instead of considering the most general quadric, the procedure or algorithm for determining the parameters which define the surface from limb images coordinates may be illustrated by considering the simplest case, *viz.*, the sphere. For this case, four parameters must be determined: r, the sphere's radius and, $\mathbf{h}$, the sphere's center coordinates. Each measured limb image coordinate pair yields an equation in these unknowns; but, because the equation is non-linear, an initial guess of the four unknowns must be supplied. The algorithm, which is simply Newton's method, will now be given.

(1) Given **A, a** (first picture camera model)
B, b (second picture camera model)

and measured limb image coordinates on the first picture

(u_1, v_1)
.
.
.
(u_{n_1}, v_{n_1})

and limb image coordinates on the second picture

(x_1, y_1)
.
.
.
(x_{n_2}, y_{n_2})

with $n_1 + n_2 \geqslant 4$, $n_1 \neq 0$, and $n_2 \neq 0$. Also given are $\mathbf{h}_0$ and $\mathbf{R}_0$, initial guesses of the sphere's center coordinates and radius squared.

(2) If utilizing measurements from the first picture,

$\mathbf{d} = \mathbf{a} + \mathbf{A}\,\mathbf{h}$ (initially, $\mathbf{h} = \mathbf{h}_o$),

$$\mathbf{Q} = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} \mathbf{A},$$

$\mathbf{M} = \mathbf{Q}\,\mathbf{Q}^{\mathrm{T}} - R\,\tilde{\mathbf{A}}\,\tilde{\mathbf{A}}^{\mathrm{T}}$ ($\tilde{\mathbf{A}} = \mathrm{cof}\,\mathbf{A}$, $R = r^2$, initially $R = R_o$),

$\mathbf{P} = \mathbf{Q}\,\mathbf{A}^{\mathrm{T}}$,

$$\frac{\partial \mathbf{M}}{\partial d_1} = \begin{pmatrix} 0 & -P_{13} & P_{12} \\ & -2P_{23} & P_{22} - P_{33} \\ & & 2P_{32} \end{pmatrix},$$

$$\frac{\partial \mathbf{M}}{\partial d_2} = \begin{pmatrix} 2P_{13} & P_{23} & P_{33} - P_{11} \\ & 0 & -P_{21} \\ & & -2P_{31} \end{pmatrix},$$

$$\frac{\partial \mathbf{M}}{\partial d_3} = \begin{pmatrix} -2P_{12} & P_{22} - P_{33} & -P_{32} \\ & 2P_{21} & P_{31} \\ & & 0 \end{pmatrix},$$

$$\frac{\partial \mathbf{M}}{\partial h_i} = A_{i1}\frac{\partial \mathbf{M}}{\partial d_1} + A_{i2}\frac{\partial \mathbf{M}}{\partial d_2} + A_{i3}\frac{\partial \mathbf{M}}{\partial d_3}, \text{ and}$$

$$\frac{\partial \mathbf{M}}{\partial R} = -\tilde{\mathbf{A}}\tilde{\mathbf{A}}^{\mathrm{T}}.$$

(3) Given (u_i, v_i) measured coordinates on image limb, compute

$\mathbf{u}^{\mathrm{T}}_i = (u_i, v_i, 1)$,

$f_i = \mathbf{u}^{\mathrm{T}}_i\,\mathbf{M}\,\mathbf{u}_i$, and

$$J_i = \left(\mathbf{u}_i^{\mathrm{T}} \frac{\partial \mathbf{M}}{\partial h_1} \mathbf{u}_i,\;\; \mathbf{u}_i^{\mathrm{T}} \frac{\partial \mathbf{M}}{\partial h_2} \mathbf{u}_i,\;\; \mathbf{u}_i^{\mathrm{T}} \frac{\partial \mathbf{M}}{\partial h_3} \mathbf{u}_i,\;\; \mathbf{u}_i^{\mathrm{T}} \frac{\partial \mathbf{M}}{\partial R} \mathbf{u}_i \right).$$

(4) Form the $n_1 \times 1$ column vector and the $n_1 \times 4$ matrix

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{n_1} \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} J_1 \\ J_2 \\ \cdot \\ \cdot \\ \cdot \\ J_{n_1} \end{pmatrix}.$$

(5) Repeat steps (2), (3), and (4) with second picture parameters to form

$\mathbf{g}$ and $\mathbf{G}$.

(6) Form an improved estimate of $\mathbf{h}$ and $\mathbf{R}$ as

$$\begin{pmatrix} \mathbf{h}_{i+1} \\ R_{i+1} \end{pmatrix} = \begin{pmatrix} \mathbf{h}_i \\ \mathbf{R}_i \end{pmatrix} - \begin{pmatrix} \mathbf{J} \\ \mathbf{G} \end{pmatrix}^{\mathrm{I}} \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}.$$

where $\mathbf{U}^{\mathrm{I}}$ is the generalized inverse of $\mathbf{U}$.

(7) Iterate until convergence.

At least four limb image coordinates must be measured, but not all measurements may be gathered from a single image. For the minimum measurement case, the accuracy with which r and $\mathbf{h}$ are found is about that with which coordinates of a point in three-space are found from image coordinates of match points. Over-determined cases yield the expected improvement in accuracy.

Termination of Quadrics by Planes

Thus far, the discussion has considered only the problem of determining parameters which define a quadric from two views of its limb image. An associated problem is that of determining parameters of a plane which truncates the quadric.

Suppose a plane intersects a quadric and the image of this intersection can be seen on one view (Figure 3). The problem is to determine parameters which define this plane. After parameters which define the quadric have been determined, plane parameters can be determined from measurements on a single image.

If image coordinates, (u,v), on the image of the curve determined by the intersect of the two surfaces, are measured the parameter, λ, of Equation 6 can be computed. For

$$\lambda^2\,\mathbf{u}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{u} - 2\lambda\,\mathbf{d}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{u} + \mathbf{d}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{d} = c$$

so

$$\lambda = \frac{\mathbf{d}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{u} \pm \mathbf{u}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{u}}{\mathbf{u}^{\mathrm{T}}\,\mathbf{K}\,\mathbf{u}} \qquad (16)$$

where

$\mathbf{u}^{\mathrm{T}} = (\mathbf{u}, \mathbf{v}, 1)$;

$\mathbf{K}$ and $\mathbf{d}$ are given in Equations 8 and 9, respectively; and

$\mathbf{M}$ is matrix given in Equation 10.

Thus, the three-space coordinates of the pre-image of (u, v) are given by

$$\mathbf{X} = \mathbf{A}^{-1}\,(\lambda\,\mathbf{u} - \mathbf{a}).$$

So, in general, three image coordinate pairs must be measured in order to define the truncating plane.

The sign in Equation 16 must be determined

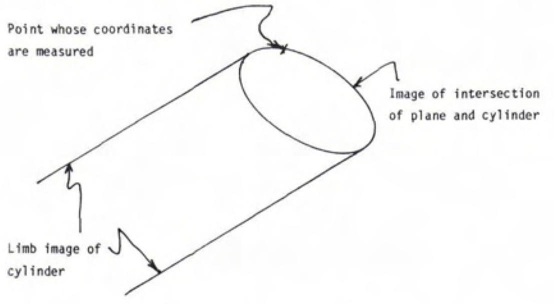

FIG. 3. Image of a truncated cylinder.

by the operator. It is + for the point pictured in Figure 3.

ADDITIONAL COMMENTS AND SUMMARY

The limb image of any quadric may be written as

$$\mathbf{u}^T \mathbf{M} \mathbf{u} = 0$$

where $\mathbf{u}^T = (u, v, 1)$,
$\mathbf{M}$ is a 3×3 matrix which depends on parameters of the quadric and camera model, and
(u,v) are image coordinates.

The explicit form of $\mathbf{M}$ is given by Equation 10 for a non-parabolic (central) quadric and by Equation 13 for parabolic quadrics. Thus, if several pairs of image coordinates on the limb image are measured, they constrain the parameters which define the quadric. Measurements from two views of the surface are generally sufficient to define the surface in three-space (assuming the camera model of each view is known). The surface parameters enter these equations non-linearly; however, a Newton's method of solution may be used.

If the quadric is cone or cylinder (elliptic, parabolic, or hyperbolic), $\mathbf{M}$ may be factored so that $\mathbf{M} = \mathbf{S}^T \mathbf{T}$ where $\mathbf{S}$ and $\mathbf{T}$ are 3×1 vectors. Thus, the limb image of a cone or cylinder is always two straight lines; i.e.,

$$\mathbf{S}^T \mathbf{u} = 0 \text{ and}$$

$$\mathbf{T}^T \mathbf{u} = 0.$$

However, the parameters which define a general cone or cylinder cannot be found from just two views; three or more views are required. Two views are sufficient if it is known that the cone or cylinder is circular or parabolic.

Once the parameters of the quadric are found, parameters which define a truncating plane may be found from image coordinates on the image of the intersection of the plane and quadric. These image coordinates may be gathered from just a single view.

The limb itself was given little consideration here; however, it may be shown that it always lies in a plane (Fishback, 1962). The so-called "polar plane." The equation of this plane is

$$(\mathbf{N} - \mathbf{h})^T \mathbf{B}(\mathbf{X} - \mathbf{h}) = c$$

for non-parabolic quadrics, and

$$(\mathbf{N} - \mathbf{h}^T)\, \mathbf{R}^T\, \mathbf{D}\, \mathbf{R}\, (\mathbf{X} - \mathbf{h}) = \mathbf{R}_3\, (\mathbf{N} + \mathbf{X} - 2\, \mathbf{h})$$

for parabolic quadrics. Here, $\mathbf{N}$ is the coordinates of the projective center, so that

$$\mathbf{N} = -\mathbf{A}^{-1}\, \mathbf{a}.$$

Finally, if the camera model (Equation 2) can be used instead of the Equation 1 model, then $\mathbf{u}$ in Equation 8 is replaced by $\mathbf{u}^T = (u, v, -F)$ and $\tilde{\mathbf{A}}$ of Equations 10 and 13 is replaced by $\mathbf{A}$ (because $\mathbf{A}^T = \mathbf{A}^{-1}$ implies $\tilde{\mathbf{A}} = \mathbf{A}$).

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