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What Is a Near-Vertical Photograph?

When are aerial photograph orientation angles small enough to allow the assumption of a linearized rotation matrix?

(Abstract on next page)

INTRODUCTION

IF ALL AERIAL PHOTOGRAPHS were truly vertical, with a photo exposure plane parallel to the object datum plane, the mathematics of metric photogrammetry would be relatively simple: the parallax equations could be used to reduce photo measurements to object coordinates. Unfortunately, they are seldom vertical, but rather are nearly always exposed with the camera axis tilted slightly from the vertical. This fact requires that the angular orientation of each exposure be considered, and it introduces these angles into the mathematics of metric photogrammetry.

The most fundamental relationships between object coordinates and the image coordinates of points on an aerial photograph are the collinearity equations, namely;

where (ω, ϕ, κ) are the angles of rotation which represent the tilt of the exposure plane about the axes of the photograph (see Section 2.2.3.2.2, *Manual of Photogrammetry* (Slama, 1980)).

The collinearity equations are much used in analytical photogrammetry. Unfortunately, they are nonlinear. This nonlinearity, and the trigonometric functions, adds to the complexity of the computational procedures and thus leads some analysts to consider simplification of their form (Jeyapalan, 1983). The most direct simplification is provided by a linearization of the rotation matrix. This requires a "small angles" restriction on the magnitude of ω, ϕ, and κ; a restriction satisfied apparently by "near-vertical" aerial photographs. The objective of this paper is to quantify the meaning of the term "near-vertical."

$$x_j - x_p + \frac{f[m_{11}(XJ - XL) + m_{12}(YJ - YL) + m_{13}(ZJ - ZL)]}{[m_{13}(XJ - XL) + m_{32}(YJ - YL) + m_{33}(ZJ - ZL)]} = 0,$$

$$y_j - y_p + \frac{f[m_{21}(XJ - XL) + m_{22}(YJ - YL) + m_{23}(ZJ - ZL)]}{[m_{31}(XJ - XL) + m_{32}(YJ - YL) + m_{33}(ZJ - ZL)]} = 0.$$

where (x_j, y_j) denote the photo coordinates of image point j; (x_p, y_p) are the photo coordinates of the principal point; f is the camera focal length; (XJ, YJ, ZJ) are the object coordinates of point j; and (XL, YL, ZL) are the object coordinates of the photograph exposure center. (Note: The terminology used here resembles that of the *Manual of Photogrammetry* (Slama, 1980). The collinearity equations appear there as Equations (2.234).)

The m_{ij} terms in these equations are the direction cosines of the rotation matrix which relate the object datum plane and the tilted photograph exposure plane. These have the values, in the usual matrix format, of

$$\begin{matrix} \cos\phi\cos\kappa & \sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa & -\cos\omega\sin\phi\cos\kappa + \sin\omega\sin\kappa \\ -\cos\phi\sin\kappa & -\sin\omega\sin\phi\sin\kappa + \cos\omega\cos\kappa & \cos\omega\sin\phi\sin\kappa + \sin\omega\cos\kappa \\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi \end{matrix}$$

THE LINEARIZED ROTATION MATRIX

If one assumes a near-vertical photograph and the associated small angles, one can replace the rotation matrix with the first-order approximations to the trigonometric functions, namely,

$$\begin{matrix} 1 & \kappa & -\phi \\ -\kappa & 1 & \omega \\ \phi & -\omega & 1 \end{matrix}$$

This matrix simplifies the mathematics; however, it obviously introduces large errors for large angles. Our question, in the context of analytical photogrammetry, should be, "What are large angles?"

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ABSTRACT: *The photogrammetric literature has reported on several simplified analytical plotters. These instruments are generally not first-order but rather have limited the quality of optics, the servo-systems, the photo-measurement precision, and the computer hardware and/or software support. In addition, some have restricted application to "near-vertical" aerial mapping photography in order to justify the introduction of the linearized rotation matrix into the image-to-object collinearity equations and thereby simplify the descriptive mathematics. This paper shows this to be an impractical limitation. A survey of Norwegian aerial mapping photography, considered typical of world standards, showed that exposure-plane orientation angles are commonly of 1 to 2 gons (0.9° to 1.8°) in magnitude. Such small angles can introduce the equivalent of several hundred micrometres in photo-measurement errors if a lineraized rotation matrix is used. This study concludes that, in the context of analytical photogrammetry and current technology, there are very few "near-vertical" photographs available for aerial mapping.*

ERROR INTRODUCED BY LARGE ANGLES

The values for exposure plane image coordinates (x_j, y_j) can be computed directly with the collinearity equations. Values can be computed with either the full or the simplified rotation matrix; both sets of coordinates are functions of the same variables. The difference between the two sets of coordinates can be used as a measure of the error introduced by the linearized rotation matrix. Of course, with ten variables a problem of display arises: how to present the error variation in compact form? One form is suggested here.

The collinearity equations can be simplified for display by normalizing the object coordinates with the flying height above datum, ZL. One can also remove the arbitrary influence of (x_p, y_p) and (XL, YL) by taking their value as zero. These steps leave the forms

$$x_j = -f\frac{m_{11}(XJ/ZL) + m_{12}(YJ/ZL) + m_{13}(ZJ/ZL - 1)}{m_{31}(XJ/ZL) + m_{32}(YJ/ZL) + m_{33}(ZJ/ZL - 1)}$$

and

$$y_j = -f\frac{m_{21}(XJ/ZL) + m_{22}(YJ/ZL) + m_{23}(ZJ/ZL - 1)}{m_{31}(XJ/ZL) + m_{32}(YJ/ZL) + m_{33}(ZJ/ZL - 1)}$$

In these equations XJ/ZL is bounded approximately by the camera film-size/focal-length ratio; in other words, the camera angle of coverage. The same is true for YJ/ZL. The ratio ZJ/ZL is related to the ground relief: in mountainous country the value could be relatively large, perhaps 0.2 to 0.4; over flat country it could be taken as zero.

The error in the computed photo-coordinates due to use of the simplified rotation matrix will, of course, vary with the magnitude of XJ/ZL, YJ/ZL, and ZJ/ZL as well as the rotation angles. For display, one can let particular $(XJ/ZL, YJ/ZL)$ coordinates represent a region or cell on the exposure plane where errors will be shown. Figure 1 shows a choice of six such cells: they cover regions on a photograph—the left of a stereopair—where one could expect image points of interest to appear.

Figure 1 is a key that should be used to relate the error magnitudes, shown in Tables 1 and 2, to an approximate position on an exposure plane. These tables are compiled to show the influence of angles ω and ϕ upon the error of computed image point coordinates when the image point occurs, generally, in a particular region of the photograph. For example, object points with normalized $(XJ/ZL, YJ/ZL)$ coordinates of (2/3, 2/3) will have image points in the upper-right corner of the photograph: cell 2 in all tables. The array of numbers in cell 2 shows the errors, in micrometres, for combinations of ω and ϕ, each at angles of -2, -1, 0, 1, or 2 gons ($-1.8°$, $-0.9°$, 0°, 0.9°, or 1.8°). The reader will note that the two tables show the same blocks of cells, one relevant to the photo x-coordinate and one to the y-coordinate.

The tables are distinguished by fixed values for the angle κ, the parameter ZJ/ZL, the camera focal length, and whether they apply to the photo x- or y-coordinate. The tables published here are rather specific; they are presented only as examples for discussion. The computation of other examples should be quite easy for anyone with access to a digital computer.

DISCUSSION OF ERRORS

The most remarkable thing about the errors

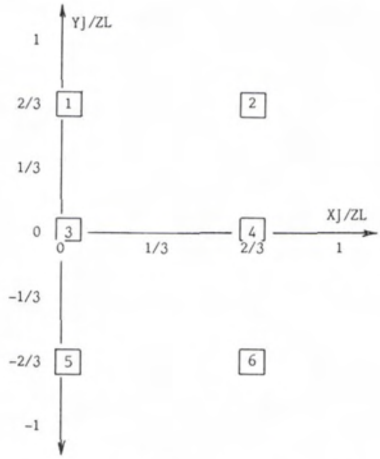

FIG. 1. Numbered "Cells" on an exposure plane identifying regions where image points of interest might appear.

shown in Tables 1 and 2 are their size: several hundred micrometres for angles of the order of 2 gons (1.8°). This is particularly remarkable in the context of analytical photogrammetry where instruments are being built with encoder resolutions of 5 micrometres or better, and where inclusion of the full rotation matrix adds very few difficulties or computation time. In the face of such errors, one must question whether there are any aerial photographs, exposed under practical conditions, that can be considered near enough to vertical to justify the use of the linearized rotation matrix.

Orientation Angles in Practical Aerial Mapping Photography—A Norwegian Example

Norway has a national aerial mapping program, and every effort is made to provide truly vertical aerial photographs to its mapping offices. Of course, as the *Manual of Photogrammetry* states, "truly vertical aerial photographs must be considered a fortunate accident," in the face of practical conditions. The records of one Norwegian mapping office were examined to provide an example of how near the "truly-vertical" objective a controlled, yet practical effort generally comes.

The Norwegian Land Reallocation Department, Photogrammetric Office, constructs contour and cadastral maps. They use the national supply of aerial mapping photographs; generally 1:15,000-scale photographs exposed through registered, calibrated, wide-angle frame cameras with focal lengths near 150 millimetres. Among the plotting machines used by this office are a Wild Autograph A8, a B8S, and a Zeiss Planicomp C-100 Analytical Stereoplotter. Records are kept of the orientation angles of stereomodels mounted in these machines. These many hundreds of records show what orientations can be expected in practical aerial mapping photography.

A random sample of 100 orientation reports was taken from the records. From each report three orientation angles were computed as the difference between the angles in the stereopair. The averages of the absolute value of these differences were 0.74, 0.68, and 1.07 gons (0.67°, 0.61°, 0.97°) for angles ω, ϕ, and κ, respectively. The standard deviations of these angles about zero were 1.06, 0.93, and 1.56 gons (0.95°, 0.84°, 1.40°), respectively. It was noted

TABLE 1. ERROR IN PHOTO *X*-COORDINATE, MICROMETRES (FOR κ = 2.0, ZJ/ZL = 0.20, FOCAL LENGTH = 150 MM.)

	Cell 1					Cell 2				
ϕ/ω	−2.0	−1.0	0.0	1.0	2.0	−2.0	−1.0	0.0	1.0	2.0
−2.00	280	139	2	−131	−262	271	89	−59	−174	−259
−1.00	216	107	0	−102	−203	213	59	−60	−146	−203
0.00	151	74	0	−73	−144	152	27	−62	−119	−145
1.00	87	42	−1	−43	−85	91	−3	−63	−90	−86
2.00	26	13	0	−11	−23	31	−32	−61	−57	−22
	Cell 3					Cell 4				
ϕ/ω	−2.0	−1.0	0.0	1.0	2.0	−2.0	−1.0	0.0	1.0	2.0
−2.00	148	74	0	−73	−147	142	26	−60	−117	−146
−1.00	149	74	0	−73	−147	146	28	−60	−118	−146
0.00	148	74	0	−74	−148	148	27	−61	−120	−148
1.00	147	73	0	−74	−149	150	27	−63	−122	−150
2.00	147	73	0	−74	−148	154	29	−62	−122	−149
	Cell 5					Cell 6				
ϕ/ω	−2.0	−1.0	0.0	1.0	2.0	−2.0	−1.0	0.0	1.0	2.0
−2.00	23	11	0	−13	−26	19	−34	−61	−59	−26
−1.00	85	43	1	−42	−87	82	−2	−59	−88	−86
0.00	144	73	0	−74	−151	143	27	−61	−121	−150
1.00	203	102	0	−107	−216	206	58	−63	−155	−217
2.00	262	131	−2	−139	−280	271	89	−64	−189	−284

TABLE 2. ERROR IN PHOTO y-COORDINATE, MICRONMETRES (FOR κ = 2.0, ZJ/ZL = 0.20, FOCAL LENGTH = 150 MM.)

	Cell 1					Cell 2				
φ/ω	−2.0	−1.0	0.0	1.0	2.0	−2.0	−1.0	0.0	1.0	2.0
−2.00	150	148	148	147	145	142	143	143	142	138
−1.00	27	26	27	28	28	24	26	27	28	25
0.00	−62	−63	−61	−60	−60	−64	−63	−61	−59	−61
1.00	−120	−121	−120	−119	−119	−122	−122	−121	−120	−121
2.00	−145	−147	−148	−148	−150	−147	−149	−150	−150	−153
	Cell 3					Cell 4				
φ/ω	−2.0	−1.0	0.0	1.0	2.0	−2.0	−1.0	0.0	1.0	2.0
−2.00	149	148	148	147	146	144	144	144	144	141
−1.00	73	73	74	74	74	71	72	73	73	72
0.00	0	0	0	0	0	−2	0	0	1	0
1.00	−74	−74	−74	−73	−73	−76	−75	−74	−74	−76
2.00	−146	−147	−148	−148	−149	−151	−151	−151	−152	−155
	Cell 5					Cell 6				
φ/ω	−2.0	−1.0	0.0	1.0	2.0	−2.0	−1.0	0.0	1.0	2.0
−2.00	150	148	148	147	145	148	146	145	145	143
−1.00	119	119	120	121	120	118	118	119	120	118
0.00	60	60	61	63	62	59	60	62	63	61
1.00	−28	−28	−27	−26	−27	−30	−29	−27	−27	−30
2.00	−145	−147	−148	−148	−150	−153	−152	−152	−153	−159

further that, among the 100 stereopairs, 19 involved at least one angle greater than 2 gons (1.8°) and seven showed angles over 3 gons (2.7°).

CONCLUSIONS

As analytical photogrammetric systems become more common, many mapping offices are likely to consider simplified analytical image space plotters for map revisions, thematic mapping, or other plotting tasks based upon metric photogrammetry. Several such instruments have been mentioned in the literature: the Zeiss Stereocord, the Analytical Point Positioning System (Konecny, 1980), and the SDP System (Jeyapalan, 1983). Each of these has reportedly based their collinearity equations upon the linearized rotation matrix and, therefore, has restricted applications to the use of "near-vertical" aerial mapping photography. Based upon this study, this restriction seems impractical.

More specifically, if an analytical instrument is built to encode photo coordinates to a precision of better than fifty micrometres, it must not restrict its collinearity equations with the linearized rotation matrix. This would be inconsistent because, as this simple study has shown, the errors introduced by the small angles of practical aerial mapping photography are greater than the photo measurement error introduced by a practical encoder system.

In general, it must be concluded that, in the context of analytical photogrammetry and current technology, there are very few "near-vertical" aerial mapping photographs.

REFERENCES

Jeyapalan, K., 1983. A Layman's Analytical Plotter: The SDP System. *Photogrammetric Engineering and Remote Sensing*, Vol. 49, No. 11, pp. 1569-1575.

Konecny, G., 1980. How the Analytical Plotter Works and Differs from an Analog Plotter. *Proceedings of the Analytical Plotter Symposium and Workshop*, American Society of Photogrammetry, ISBN 0-937294-00-4, pp. 31-75.

Slama, C. (ed.), 1980. *Manual of Photogrammetry*, 4th Edition, American Society of Photogrammetry, Falls Church, Va., Chapters II and V.

Thompson, E. H., 1968. Corrections to x-Parallaxes, *The Photogrammetric Record*, Vol. 6, No. 32, pp. 202-210.

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