# Vertical Exaggeration in Stereo-Vision: Theories and Facts

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ABSTRACT: Several hypotheses and mathematical formulas have been proposed to explain and quantify the phenomenon of vertical exaggeration in stereo vision. Initially, stereoscopic images were supposed to be formed by convergency of visual axes at the so called "plane of fusion." However, this convergency theory failed to explain stero vision when visual axes are parallel or divergent. Then, the hypothesis of a geometric image formed at the plane of fusion was replaced by that of a perceptual image formed at the "virtual fixation plane" in a non-geometric perceptual space. This new approach has strongly influenced later investigations, up to the point that most of the proposed formulations include the distance to the perceptual image as a determinant factor of vertical exaggeration. In general, the investigations have not been free from inconsistencies between theories and facts, some of which are discussed in this paper.

## INTRODUCTION

THE SEARCH for a mathematical equation to determine the vertical exaggeration in stereo vision has been considered of great importance in photointerpretation and other stereoscopy applications. Unfortunately, none of the proposed formulations has become convincing enough to receive broad acceptance. It may be that, after so many failed efforts in the search for a solution to this problem, the topic of vertical exaggeration had lost interest, despite that, in theory, the solution is logically feasible. It is this logical but unknown feasibility what makes the topic particularly important, both scientifically and technologically. Evidently, the possibility of obtaining precise control of depth perception in stereo vision would broaden the field of technological applications, not only in the handling of photographic stereopairs but also in creating stereopairs of drawings for producing accurate representations of objects or phenomena in three dimensions. In addition, more precise instruments for direct stereo vision of tridimensional objects, with absolute control of depth perception, could be designed. The purpose of the present paper is to show, for the first time, some inconsistencies that have affected most of the studies on vertical exaggeration, which have given rise to incorrect statements used as premises by several authors. It is also an attempt to synthesize the real status of the investigations in this field, as a conceptual basis which permits the unification of concepts within the controversy that has characterized the hypotheses proposed on vertical exaggeration.

## GENERAL APPROACH

Through a rapid review of the investigations concerning vertical exaggeration, it was observed that two basic methodologies were used: induction and deduction.

- *Induction.* This method has been followed by Thurrell (1953) and Miller (1953). They started from empirical observations by using stereoscopic photographs of plaster of paris blocks. Photographic variables, as well as block dimensions, were compared with the perceived vertical exaggeration. Although these authors did not find precise relationships between vertical exaggeration and other optical variables, they succeeded in visualizing some qualitative connections. Stone (1951, p. 757) induced from his experiences a formula for determining vertical exaggeration. Such a formula is rather the result of an intuitive feeling obtained by that author in the use of Production and Marketing Administration photos of the United States. It is possible that this formula will give satisfactory results in some particular cases such as those handled by its author, but it is improbable that its validity can be extended to other cases. A similar opinion is expressed by Treece (1955, p. 519).
- *Deduction.* This method has been followed by most of those who have proposed mathematical formulas for determining vertical exaggeration (Goodale, 1953; Raasveldt, 1956; Miller, 1958; Yacoumelos, 1972; La Prade, 1972; Collins, 1981). Starting from premises based on personal hypotheses, each author deduces a different formula for determining vertical exaggeration. However, experience has shown that those formulas are valid for only a small range of values, or are applicable with too many reservations, or are frankly unreliable. Later in this paper it will be shown that this lack of reliability does not originate in operational errors but in inconsistencies between statements and facts.

## EVOLUTION OF THEORIES

Different theories have been postulated regarding the characteristics of the spatial image which is perceived when a pair of overlapping photographs is

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viewed stereoscopically. Initially, attention was focused on the spatial image which is formed by convergency of visual axes in the so called "plane of fusion." It was supposed that this geometric image would correspond to the mentally perceived one. On this conceptual basis, Aschenbrenner (1952) proposed a formula for determining vertical exaggeration (Table 1). However, facts have shown that the perceived image usually appears distorted with respect to the geometric image, which invalidates the theory of convergency. In addition, the theory of convergency does not explain the fact that stereoscopic vision can also be achieved with parallel or divergent visual axes, in which case no intersection of visual axes takes place and consequently no image is geometrically yielded.

Goodale (1953) proposed the hypothesis that the perceived image was equivalent to the one formed by intersection of optical rays coming both from the eyes and from two positions of the air-camera lens, to homologous points on the photographic images located at the focal distance. This hypothesis made it possible to construct geometric images even under conditions of parallelism or divergency of visual axes. From the corresponding diagrams, Goodale (1953, p. 615) derives a mathematical formula for determining vertical exaggeration (Table 1) which he claims to have been proved experimentally. However, after his equation was questioned by Treece (1955, p. 521), Goodale (1955, p. 527) recognizes that "In my opinion, we have had enough theorizing. What is needed now are good, sound experimentation and proofs."

Raasveldt (1956, p. 710) formulates another hypothesis : In his opinion, "we do not 'see' the outside physical world but our mentally projected impressions." These impressions, in turn, are conditioned to some visual telemetric proprieties that are regulated physiologically by the degree of tension of certain muscles, such as the recti ones, that control the angularity of visual axes producing the feeling of convergency, and the cilliary ones, that regulate the accomodation to visual distance. Raasveldt suggests that, due to the above mentioned telemetric proprieties, or principles, the stereoscopic model is observed at a certain distance, different from that of the intersection of optic rays. So, even when visual axes are parallel or divergent, and no real plane of fusion is present, the streights of the telemetric principles allow one to observe a distinct model at a certain place called the "virtual fixation point."

The concept of the "virtual fixation point" envolves the notion of "perceptual space" where stereoscopic images are perceived, and establishes a difference between geometric image and perceptual image. These ideas have influenced notably the subsequent development of theories concerning vertical exaggeration. As a result, most mathematical formulas, such as those of Raasveldt, Miller, La Prade, and Collins (Table 1), include, as a determinant variable, the distance to the perceptual image. The problem is that such a distance has to be obtained by visual estimation, and this operation is not free from subjective inferences.

Table 1. Mathematical Equations Proposed for Determining Vertical Exaggeration.

| | | |
|---|---|---|
| (1) | STONE (1951, p. 757) | $E = \frac{CF}{ef}$ |
| (2) | ASCHENBRENNER (1952, p. 821, eq. 3a) | $E = \frac{BD}{He}$ |
| (3) | GOODALE (1953, p. 610) | $E = \frac{D(e + S)(b + p)}{Fe(e + mp)}$ |
| | When $S = e$ (p. 611) | $E = \frac{2D(b + p)}{F(e + mp)}$ |
| (4) | RAASVELDT (1956, p. 721, eq. 6) | $E = \frac{Lb}{eF}$ |
| (5) | MILLER (1958, p. 813) | $E = \frac{LB}{eH} = \frac{LP}{eF}$ |
| | Approximated (p. 814): | $E = \frac{16P}{eF}$ |
| (6) | YACOUMELOS (1972, p.796, eq. 4.6) | $E = \frac{ef}{Fs}$ |
| | With magnification (eq.4.10) | $E = \frac{f}{Fm}$ |
| (7) | LA PRADE (1972, p. 1185) | $E = \frac{BL}{He}$ |
| | Approximated: | $E = \frac{5B}{H}$ |
| (8) | COLLINS (1981, p. 49, eq. 20) | $E = \frac{1}{De^{1/2}}\left(\frac{B}{H}\right)^{1/2} L$ |

STANDARDIZED CONVENTIONS

$B$ = camera base (air base)
$H$ = camera height (flying height)
$F$ = camera focal length
$S$ = print separation
$b$ = photo base
$c$ = picture edge distance, measured along the line of flight between the visible edges of two overlapping photos.
$P$ = absolute parallax
$p$ = image displacement (parallax difference)
$s$ = stereoscope base
$f$ = stereoscope focal length
$m$ = magnifying power of the stereoscope
$e$ = eye base
$D$ = actual viewing distance
$d$ = effective distance
$L$ = distance of virtual fixation plane (perceptual distance)

Comments: Considering that $P$ is practically equal to $b$, it can be observed that Equations 4, 5, and 7 are basically identical. In addition, these three equations are very similar to Equation 2, with the only difference being that, in Equation 2, the actual viewing distance ($D$) is used instead of the perceptual distance ($L$).

Yacoumelos (Table 1) proposed a formula for vertical exaggeration where perceptual distance is not included, thus avoiding the problem of dealing with this empirical variable. But, on the other hand, this formula has the problem of excluding also the camera base-to-height ratio, usually considered as the primary cause of vertical exaggeration.

The above outlined ideas provide the conceptual framework in which investigations have been developed until now, perhaps with more speculation than practical results.

## SOME INCONSISTENCIES

Among the investigations of vertical exaggeration, it is noticeable that statements have not always been consistent with facts. Some of these inconsistencies would require a systematic process of reasoning to be demonstrated. Others, instead are factual inconsistencies that require practically no theoretical demonstration because they are detectable through simple experimental checking.

To illustrate this situation, a few examples of factual inconsistencies, characterized by remarkable desagreements between hypotheses and facts, will be considered. They deal with three basic subjects: (1) the concept of vertical exaggeration, (2) the effect of viewing distance upon vertical exaggeration, and 3) the effect of optical magnification upon vertical exaggeration.

### Inconsistencies Regarding the Concept of Vertical Exaggeration

Vertical exaggeration is mathematically defined as the ratio of vertical scale to horizontal scale. Therefore, any study about vertical exaggeration should be developed in accordance with this fundamental premise. However, some authors have drawn their inferences so far beyond the facts, that they have misinterpreted the very meaning of vertical exaggeration, that is, the object itself of the investigation. For example, Yacoumelos (1972, p. 796), contrary to what is stated by definition, emphasizes that "we cannot speak of vertical exaggeration in mathematical terms." Miller (1953, p. 594), in turn, exhibits confusion about the mathematical meaning of vertical exaggeration, at least in so far as negative numbers are concerned as in the following quotation:

> "By moving the eyes farther from the photographs, a positively exaggerated image of the model would be obtained (Figure 1b: Vertical Exaggeration 1.50), and by decreasing eye-to-photograph distance, a negative exaggeration would be produced (Figure 1c: Vertical Exaggeration 0.65)."

In the above quotation, the value 0.65, perhaps because it is less than 1, is erroneously considered as a negative exaggeration.

Finally, Yacoumelos (1973, p. 274), probably in view of so many failed efforts in the search for a mathematical expression of vertical exaggeration, comes to the point of denying the existence of vertical exaggeration, and consequently discards the possibility of measuring it, in the following terms:

> "Still the fact remains that to continue the search for a mathematical expression that will give a quantitative measure of what it is called (although it does not exist) vertical exaggeration is a Quixotic effort."

### Inconsistencies Regarding the Effect of Viewing Distance upon Vertical Exaggeration

It is not necessary to go into deep analyses to prove that, when a pair of overlaping photographs is viewed stereoscopically, the vertical exaggeration of the perceived image increases with viewing distance. The mere experience of viewing photographic stereopairs with the naked eye at different distances allows one to verify this fact. However, in spite of this evidence, some authors have denied that vertical exaggeration increases with viewing distance.

One of those authors is Salzman (1950) who shows graphically that, inasmuch as depth perception is caused by retinal disparity, and this disparity diminishes when viewing distance is increased, it can be concluded that depth perception also diminishes with viewing distance. This conclusion is correct. The error consists in identifying the idea of depth perception with that of "apparent relief." In this way, it is equally concluded that the apparent relief should not be exaggerated when viewing distance is increased. A logical explanation of this fallacy is presented in the following quotation from Miller (1953, p. 601):

> "There is some disagreement as to how the vertical dimension of the model changes when this viewing distance is changed. Some state that with increased viewing distance there is an increase in vertical exaggeration. On the other hand, there are those who maintain that changes in viewing distance have no such effect. Salzman (1950), for example, has taken this latter view. The author believes that there is very definitely a direct relation between viewing distance and vertical exaggeration. With increased distance both vertical and horizontal model dimensions change, and the ratio of vertical scale to horizontal scale, which is the expression of vertical exaggeration, is increased."

In other formulas, such as those by Raasveldt (1956), Miller (1958), La Prade (1972), and Collins (1981), vertical exaggeration is not expressed in terms of the viewing distance but as a function of the perceptual distance, which has to be obtained by visual estimation (Table 1). In the author's opinion, this visual estimation deals mainly with subjective inferences rather than with real objective features and, therefore, the data so obtained seems quite unreliable. Nevertheless, in these formulas, it could be understood, at least theoretically, that the viewing distance is implicitly contained in the perceptual distance.

La Prade (1972) neglects the influence of viewing distance upon vertical exaggeration by proposing a simplified formula in which vertical exaggeration is approximately equal to five times the base to height ratio of camera stations (Table 1).

### Inconsistencies Regarding the Effect of Optical Magnification upon Vertical Exaggeration

Another unquestionable fact is that *vertical exaggeration is a decreasing function of optical magnification*. Simple experience shows that, when a stereoscopic pair of photographs is viewed under different degrees of magnification, the vertical exaggeration varies inversely with magnification. It is not a matter of opinion but a question of fact. However, contrary to this evidence, some authors (Salzman, 1950) have maintained that vertical exaggeration does not varie inversely but directly with magnification. This idea is expressed in the following part of his paper (p. 477):

> "There are only two basic ways of exaggerating apparent relief. One is by increasing the air base distance when accomplishing photography, and the other is by enlargement of the photographic images."

Thurrell (1953, p. 581) falls into the same error when he describes his observations on magnification:

> "Magnification. Vertical exaggeration varies directly with magnification. The quantitative effect has not been analyzed because the pocket folding type stereoscope with four-inch, fixed focal length, two-power lenses was used in the preponderance of tests. This magnification was treated as a constant in these experiments."

In the quoted paragraph, it is not easy to understand how Thurrell was able to deduce that "vertical exaggeration varies directly with magnification" if, according to his words, "magnification was treated as a constant in these experiments."

On the other hand, some other authors (Raasveldt, 1956; Miller, 1958; La Prade, 1972) have stated, despite the facts, that optical magnification has no effect upon vertical exaggeration, as shown in their formulas (Table 1). To sustain their thesis, they appeal to the fallacious argument that, inasmuch as the magnifying effect is simply to multiply vertical and horizontal values by the same factor, vertical exaggeration is not affected (Raasveldt, 1956, p. 721; Miller, 1958, p. 813; La Prade, 1972, p. 1185). Without this argument, Aschenbrenner (1952) excludes also the magnifying factor from his formula (Table 1).

In a more implicit way, Collins (1981) shares the erroneous concept of Salzman and Thurrell according to which vertical exaggeration does not vary inversely but directly with magnification. In Collins' formula (1981, p. 49, eq. 20), vertical exaggeration varies inversely with $L$ which in its turn varies inversely with $w$ (image width) (1981, p. 49, eq. 12). Hence, vertical exaggeration would vary directly with image width, just the opposite of what is shown by facts.

## CONCLUSIONS

- Among the several theories and mathematical formulations that have been postulated for explaining and quantifying the phenomenon of vertical exaggeration, there is a remarkable lack of conceptual unity and a consequent high level of controversy. This is indicative of the low degree of consistency and reliability obtained up to now among the investigations.
- In the development of investigations, inductive and deductive methods have been applied independently rather than simultaneously as would be most advisable. Some authors have been involved in trying to induce conclusions from simple experimental data without an imaginative preconception of the problem. Others, on the contrary, have directed their efforts to postulate hypotheses about what each of them imagines may be the mechanism of perception of stereoscopic images, without the necessary experimental checking. Both inductive and deductive methods would better be considered as different but complementary viewpoints in the solution of a problem.
- Every proposed mathematical formula has been reported to agree with results of observations made with pocket or mirror stereoscopes. This means that variables depending from viewing conditions, such as optical magnification, viewing distance, print separation, etc., were restricted to the standarized specifications of those instruments, and that the corresponding formulas were checked only for a very small range of values. No author has intended to go farther in analyzing, for example, the case when optical magnification, instead of being 2× or 2.5× as in common stereoscopes, would be of 100× or more; or the case when the magnification factor become less than 1, in which case image reduction instead of enlargement takes place.
- The fact of limiting the field of observations within the standards of common stereoscopes has impeded authors from detecting the real importance of some variables such as optical magnification, which was neglected in most of the proposed formulations.
- Other variables that do not depend on the restricting specifications of viewing instruments, such as base ($B$) and height ($H$) of camera stations, are more susceptible to be varied experimentally. This circumstance has made it easier to visualize the influence of these two variables upon vertical exaggeration and, consequently, to reach one of the few points of agreement : that there is a strong direct correlation between vertical exaggeration and the $B/H$ ratio. However, this striking evidence has created a general tendency to overestimate the influence of the $B/H$ ratio. More than one author (Miller, 1958; La Prade, 1972) have proposed formulas in which the $B/H$ ratio becomes the only variable effecting the vertical exaggeration. It may also help to explain the consequent propensity to minimize and even neglect other important variables, as mentioned in the above item. A quite exceptional case is the formula proposed by Yacoumelos (1972) where the $B/H$ ratio is not considered as a determinant factor of vertical exaggeration. (Table 1).
- Most proposed mathematical formulas, such as those of Raasveldt (1956, p. 721), Miller (1958, p. 813), La Prade (1972, p. 1185), and Collins (1981, p. 49), in-

clude, as necessary data for calculating vertical exaggeration, the perceptual distance, which has to be visually estimated (Table 1). This implys that the calculation of vertical exaggeration has to depend on stereoscopic observations whose results cannot be logically infered beforehand. In other words, these formulas are essentially empirical. They are conceived to measure the vertical exaggeration of a given image, but not to make quantitative inferences about its mathematical relations with other variables. For example, none of the above mentioned formulas would permit one to infer the way that viewing distance, camera focal length, base of photographs, and other variables could be conveniently fixed in order to obtain a desired vertical exaggeration.

- In the author's opinion, the estimation of perceptual distance, which is required to solve the formulas mentioned in the above item, obeys mainly subjective inferences. Therefore, he discards the possibility of obtaining a reliable measurement of the perceptual distance.
- Despite the many intriguing questions that the study of the vertical exaggeration effect has offered to investigators, no tangible progress has been achieved in this field. Such a prolonged stagnation of the investigations has created an atmosphere of skepticism with respect to future developments. In 1973, Jacoumelos (p. 274) doubted the existence of vertical exaggeration itself, and considered the search for a mathematical expression to measure it as a "Quixotic effort." The author disagrees with this latter point of view. In his opinion, the search for a reliable formula to determine vertical exaggeration, rather than being a Quixotic effort, represents a quite logical and feasible task. Presently, the author is preparing a paper about new features concerning the stereo vision phenomenon, which he feels will lead to a final solution of the vertical exaggeration problem.

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## REFERENCES

Aschenbrenner, Claus M., 1952. A Review of Facts and Terms Concerning the Stereoscopic Effect: *Photogrammetric Engineering*, Vol. 18, No. 5, pp. 818–823.

Collins, Stanley H., 1981. Stereoscopic Depth Perception: *Photogrammetric Engineering*, Vol. 47, No. 1, pp. 45–52.

Goodale, E.R., 1953. An Equation for Approximating the Vertical Exaggeration of a Stereoscopic View: *Photogrammetric Engineering*, Vol. 19, No. 4, pp. 607–616.

———, 1955, Discussion of Paper by Walter A. Treece: Photogrammetric Engineering, Vol. 21, No. 9, p. 527.

La Prade, George L., 1972. Stereoscopy — A More General Theory: *Photogrammetric Engineering*, Vol. 38, No. 12, pp. 1177–1187.

Miller, Charles L., 1958. The Stereoscopic Space-Image: *Photogrammetric Engineering*, Vol. 26, No. 5, pp. 810–815.

Miller, Victor C., 1953. Some Factors Causing Vertical Exaggeration and Slope Distortion on Aerial Photographs: *Photogrammetric Engineering*, Vol. 19, No. 4, pp. 592–607.

Raasveldt, Henri C., 1956. The Stereomodel, How It Is Formed and Deformed: *Photogrammetric Engineering*, Vol. 22 No. 9, pp. 708–726.

Salzman, M. H., 1950. Note on Stereoscopy: *Photogrammetric Engineering*, Vol. 16, No. 3, pp. 475–477.

Stone, Kirk H., 1951. Geographical Air-Photo-Interpretation: *Photogrammetric Engineering*, Vol. 17, No. 5, pp 754–759.

Thurrell, Robert F., Jr., 1953. Vertical Exaggeration in Stereoscopic Models: *Photogrammetric Engineering*, Vol. 19, No. 4, pp. 579–588.

Treece, Walter A., 1955. Estimation of Vertical Exaggeration In Stereoscopic Viewing of Aerial Photographs: *Photogrammetric Engineering*, Vol. 21, No. 9, pp. 518–527.

Yacommelos, Nick G., 1972. The Geometry of the Stereomodel: *Photogrammetric Engineering*, Vol. 38, No. 8, pp. 791–798.

———, 1973. Comments on Stereoscopy — A More General Theory: *Photogrammetric Engineering*, Vol 39, No. 3, p. 274.

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# Errata

In the Errata on page 832 of the June 1986 issue of *PE&RS*, which were meant to correct errors in the September 1985 issue, a further error was found. On page 1413 of the article by Malaret *et al*. (Sept. 1985 issue) Equation 6 should read

$$W_{\overline{SR}} = \Sigma_i\left[\left(\int_{\lambda_L + i\Delta\lambda}^{\hat{\lambda_L} + (i+1)\Delta\lambda} 2hc^2\lambda^{-5}\ (e^{hc/\lambda kT} - 1)^{-1} d\lambda\right)_{\overline{SR}_i}\right]$$

The miniature "cover photo" on the table of contents page of the September 1986 issue of *PE&RS* was inadvertently "mirrored" right to left.