Robust Estimation in Photogrammetry

Kurt Kubik, Dean Merchant, and *Toni Schenk*
Department of Geodetic Science and Surveying, The Ohio State University, Columbus, OH 43210-1247

ABSTRACT: This paper on photogrammetry describes the principles of robust computation methods which yield results uninfluenced by gross errors. The common characteristic of these methods is that the square of the residuals is not minimized, but another properly chosen function is minimized. The different classes of robust methods are described and compared to other methods of blunder detection.

The Danish Method, which was developed by one of the authors and today is implemented in many commercial block adjustment programs, is outlined. The problems with this method are explained and guidelines are given for its proper use. Finally, practical results with this method are summarized.

WHY NOT LEAST SQUARES

As STATED in an earlier paper (Krarup *et al.*, 1980), the method of least squares is not able to deal properly with outlyers in measurements. So, it was necessary to find other estimation principles which allow a more correct estimation under the given conditions. Robust estimation was proposed for this purpose by Huber and others (compare Hampel (1974)). Robust estimators are estimators which are relatively insensitive to limited variations in the distribution function of the measurements and, thus, to the presence of gross errors and outlyers. There exists a large class of robust estimation principles (about 70). In all of these methods one does not minimize the (weighted) square sum of the residuals,

$$\sum_{i=1}^{n} r_i^2 \rightarrow \min \; ; \; r_i \ldots \text{ residuals}$$

but properly chosen functions ϕ of r_i:

$$\sum_{i=1}^{n} \phi\,(r_i) \rightarrow \min.$$

The most well-known among these were proposed by Huber and Hampel, with $\phi(r)$ being, for Huber (1964),

$$\phi(r) = \begin{cases} r^2 \text{ if } |r| \leq 2\sigma \\ 2\sigma\,(2|r| - 2\sigma) \text{ if } |r| > 2\sigma \end{cases}$$

σ being the standard deviation of measurements, and, for Hampel (1974),

$$\sigma(r) = \begin{cases} r^2 & 0 \leq |r| < a \\ ar & a \leq |r| < b \\ \dfrac{a}{c-b}\,(cr - \dfrac{r^2}{2}) & b \leq |r| < c \\ \dfrac{a}{c-b} \cdot \dfrac{c^2}{2} & |r| \geq c \end{cases}$$

a, b, and c being constants.

Note that, in both cases, the adjustment principle depends on the magnitude of the residual r, with larger residuals contributing only little to the objective function (see Figure 1).

Another robust estimation principle is given by

$$\sum |r|^p \rightarrow \min, \; 1 \leq p < 2$$

where the most favorable range of values, p, is between 1.0 and 1.5. For $p = 1$ the estimation principle is called the least sum method (Barrodale and Young, 1966). Some of the functions which are minimized are depicted in Figure 1. Observe that in least squares outlying measurements have a large influence on the estimation, while in the robust method they have reduced influence.*

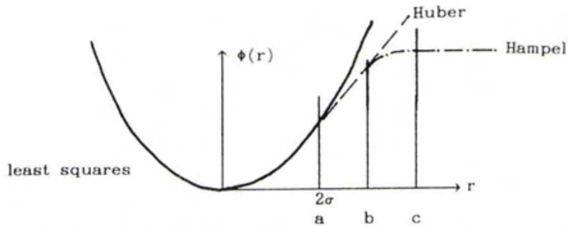

FIG. 1. Functions minimized in various adjustment principles.

The concept of robust estimation is so new that there exists no unified theory which enables us to select the best adjustment principle for our particular geodetic or photogrammetric problems. But experimentally, it was found that these methods are by far superior to least squares in the detection and location of gross errors and outlyers.

In practical computations, it is advocated that both the least-squares method and one of the alternative principles be used. A larger difference in the results indicates the presence of gross errors and requires more detailed analysis of the measurements.

The numerical solution for robust estimation consists of a repeated weighted least-squares adjustment in which the weights are adopted according to the magnitudes of the residuals of the previous iteration:

Initial weights:
$P_{i,0} = 1$, $i = 1 \ldots n$ index of measurement

weighted least squares solution in iteration α:
$\Sigma(P_\alpha\, r^2) \rightarrow \min$, $\alpha = 0,1,2,\ldots$

computation of new weights:

$$P_{i,\alpha} = \frac{\phi(r_i)}{r_i^2 + \epsilon}, \quad \epsilon \text{ small constant.}$$

or still better,

$$P_{i,\alpha} = \frac{\partial\phi/\partial r}{r_i^2 + \epsilon}$$

For the least sum method, $\Sigma|r| \rightarrow \min$, the weights are equal to $p = \dfrac{1}{|r| + \epsilon}$. The iteration stops when no significant changes in the results are observed. Usually less than 10 iterations suffice to achieve convergence.

THE DANISH METHOD

The above advantages of robust estimation methods have for some years been recognized by the Geodetic Institute of Denmark, where, since the early seventies, an automatic error search

*Most of these robust adjustment principles are derived by the maximum likelihood method from contaminated frequency distribution functions of the measurements and, thus, are optimal estimation for the assumed distribution function.

PHOTOGRAMMETRIC ENGINEERING AND REMOTE SENSING,
Vol. 53, No. 2, February 1987, pp. 167–169.
0099-1112/87/5302–167$02.25/0

routine has been used in the computation of all larger geodetic problems. This method was developed after the ideas of Krarup (personal communication) and is especially designed to eliminate outlyers in geodetic networks.

Estimation according to the Danish Method takes place according to the following iterative algorithm:

$$\Sigma(pr^2)\alpha \rightarrow \min$$
$$P_{\alpha+1} = P_\alpha \cdot f(r_\alpha)$$

$$f(r) = \begin{cases} 1 & \text{for } \dfrac{|r|\sqrt{P_0}}{m_0} < c \\ \exp - \left(\dfrac{|r|\sqrt{P_0}}{C.m_0}\right) & \text{else} \end{cases}$$

$$\alpha = 0,1,2\ldots$$

The constant C is usually set to 3, the symbol P_0 denotes the conventional weight factors, and m_0 denotes the standard deviation of the measurements.

This method was proposed by Krarup in 1967 and since then has been used as the standard computational method at the Danish Geodetic Institute for geodetic computations. During recent years, the method has also been used for other tasks by the authors.

The rate of convergence of the method seemingly depends on the conditioning of the problem and the percentage of outlyers. This percentage was found to be around 1 percent for geodetic computations. For different categories of problems, variants of the original method proved to work most efficiently. Examples are

Photogrammetric Block Adjustment
(Bundle Method (Juhl, 1980))

$$f(r) = \exp(-0.05\left(\frac{|r|\sqrt{P_0}}{i}\right)^{4.4} \quad \text{for first three iterations}$$

$$f(r) = \exp(-0.05\left(\frac{|r|\sqrt{P_0}}{m_0}\right)^{3.0} \quad \text{for further iterations}$$

Leveling Networks
(Jensen and Mark, 1980)

$$f(r) = \exp(-0.01\left(\frac{|r|\sqrt{P_0}}{m_0}\right)^{4.4} \quad \text{for first five iterations}$$

$$f(r) = \exp(-0.05\left(\frac{|r|\sqrt{P_0}}{2m_0}\right)^{20} \quad \text{for further interations}$$

Resection
(Johannsen and Kjaersgaard, 1980)

$$f(r) = \exp(-0.03\left(\frac{|r|\sqrt{P_0}}{0.4m_p}\right)^{25} \quad \text{for first three iterations}$$

$$f(r) = \exp(-0.05\left(\frac{|r|\sqrt{P_0}}{15m_p}\right)^{20} \quad \text{for following iterations}$$

m_p standard deviation of residual

These are the different weight functions in use. Common to all is the rejection of the measurements with too large residuals. Note that these cannot be classified as robust estimation methods because a non-convex function is minimized. Thus, formal proofs of uniqueness and convergence are evasive. However, according to our experience, these methods work considerably better than robust estimation in locating outlyers. Simulation tests proved existence and uniqueness of solution in a majority of cases.

WHY DOES IT WORK

Let us consider a straight line fit (Figure 2): $Y_i = a + bx_i$, $i = 1\ldots4$ with (Y_i, X_i) given data points as shown in Figure 2

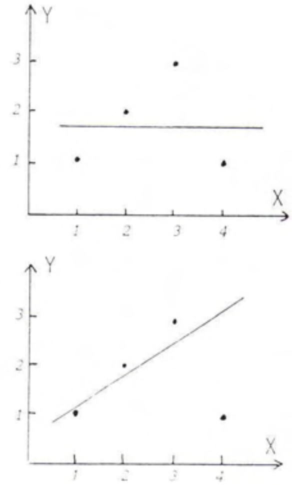

FIG. 2. (a) Least-squares fit to four data points, with one outlyer. (b) Huber's estimate of a straight line, in the presence of one outlyer, $\sigma = 0.5$.

and a, b to be estimated. In estimation, an optimal line is selected in the set of possible straight lines. In the following example, we limit this set of possible lines to ten. For these ten possibilities, the coefficients a and b and the residuals r at the four data points are listed in Table 1. Most will agree that line 7, passing through the first three data points, yields the most desirable fit ($a=0$, $b=1$). The estimation principle we select should give us this most desirable line. In Table 1 several known estimation criteria are listed, and the lines can now be classified according to these criteria. We note that line 2 is optimal in the least-squares sense. Line 7, regarded as most desirable, is optimal for the estimation criterian $\Sigma|r|^{1/2} \rightarrow \min$, $\Sigma|r|^{1/4} \rightarrow \min$, and the Danish Method. The Danish Method differentiates best between the most desirable solution and the other lines. This explains the success of robust estimation and, in particular, of the Danish Method in dealing with outlyers.

Also, an intuitive criterion for selecting line 7 as most desirable may be formalized. This line is selected because it fits most data points within a pregiven tolerance: This criterion gives, however, no unique solution. If a tolerance of 1 unit is chosen for our example, two lines classify as optimal, and an additional condition is needed to assure uniqueness. The additional conditions is

$$\Sigma r_i^2\,(\hat{a},\hat{b}) = \min\,(\Sigma r_i^2(a,b)); \text{ with } a,b \text{ unknown parameters.}$$

Thus, we select that line among the optimal ones which minimizes the residuals in a least-squares sense. The Danish Method may be interpreted as an iterative method (penalty method) for solving the above nonlinear programming problem, particularly so if the weights for outlyers are iteratively reduced to zero.

WHICH WAY TO GO

The application of robust estimation methods is still in its infancy. Much work still has to be done, both of a theoretical and practical nature. However, one thing is quite clear: the method of least squares should be used with care.

REFERENCES

Barrodale, I., and A. Young, 1966. Algorithms for Best L_1 and L_0 Linear Approximations on a Discrete Set. *Num. Mathematik*, 8, pp. 295–306.

Hampel, F.R., 1974. Modern Trends in the Theory of Robustness. *Math. Operationsforschung Statist. Ser. Statistics*, Vol. 9, No. 3, pp. 425–442.

Huber, P.J., 1964. Robust Estimation of a Location Parameters. *Ann. Math. Stat.*, 35, pp. 73–101.

TABLE 1. VARIOUS POSSIBLE SOLUTIONS IN STRAIGHT LINE FIT AND THEIR RATINGS WITH RESPECT TO DIFFERENT ESTIMATION CRITERIA.

	Possible lines through data points									
Line No.	1	2	3	4	5	6	7	8	9	10
coefficient a	1	2	3	0.5	1	1.5	0	−1	12	−3
coefficient b	0	0	0	0.5	0.5	0.5	1	1	1	1
residuals 1	0	−1	−2	0	0.5	−1	0	−1	−2	−3
2	1	0	−1	0.5	0	−0.5	0	−1	−2	−3
3	2	+1	0	1	0.5	0	0	−1	−2	−3
4	0	−1	−2	−1.5	−2	−2.5	3	+2	+1	0
estimation criteria										
$\Sigma\|r\| \rightarrow \min$	2	3	5	3	3	4	3	5	6	0
$\Sigma\|r^2\| \rightarrow \min$	5	3	9	3.5	4.5	7.5	9	7	13	27
$\Sigma\|r\|^{1/2} \rightarrow \min$	2.4	3	3.8	3	2.8	3.3	1.7	4.4	5.2	5.1
$\Sigma\|r\|^{1/4} \rightarrow \min$	2.2	3.0	3.4	2.9	2.9	3.1	1.3	4.2	4.6	3.9
Danish Method	0.9	1.1	1.5	1.0	0.9	1.8	0.4	1.6	2.0	1.3
$N(r_i \geq 0.5) \rightarrow \min$	2	3	3	3	3	3	1	4	4	3
$N(r_i \geq 1) \rightarrow \min$	2	3	3	2	1	2	1	4	4	3

N(.): number of points with .

Jensen, J., and A. Mark, 1980. *Niveauaendringer i Kobenhavn*. Graduate Report, Dept. of Photogrammetry and Surveying, Aalborg University Centre.

Johannsen and Kjaersgaard, 1980. *Afsaetning fra frie opstillinger, indordningsmetoder og beregningsmuligheder*. Graduate Report, Department of Photogrammetry and Surveying, Aalborg University Center.

Juhl, J., 1980. *Part I: Det interaktive on-line program OANA. Part II: Det analytiske Straledjævningsprogram SANA*. Thesis, Dept. of Photogrammetry and Surveying, Aalborg University Centre.

Krarup, T., K. Kubik, and J. Juhl, 1980. Götterdömmerung over Least Squares Adjustment. *Proceedings, 14th Congress of Int. Society for Photogrammetry and Remote Sensing*, Hamburg, 1980.

Rey, W.J.J., 1978. *Robust Statistical Methods*. Springer Verlag. Berlin-Heidelberg - New York.

(Received 28 April 1986; accepted 3 July 1986; revised 18 September 1986)