The Projective Equations: An Invariant Form

K. K. Rampal
Department of Civil Engineering, Indian Institute of Technology, Kanpur-208016, U. P. India

ABSTRACT: The well known projective equations of analytical photogrammetry are expressions of the relationship between the photo and ground coordinate systems and, hence, are not invariant with respect to coordinate transformation. An invariant set of equations has been derived which contains only the photo and ground distances and, hence, does not depend upon the coordinate systems of measurement, as there will be no need to transform the comparator coordinates to photo coordinates. Application of the newly derived equations has been given in the area of terrestrial photogrammetry and rectification.

INTRODUCTION

THE PROJECTIVE EQUATIONS of analytical photogrammetry, expressed by Equation 1, give the relationship between the photo and ground coordinates of point as a function of the exposure station coordinates, interior orientation elements, and exterior orientation elements: i.e.,

$$x - x_o = C\frac{a_{11}(X - X_o) + a_{12}(Y - Y_o) + a_{13}(Z - Z_o)}{a_{31}(X - X_o) + a_{32}(Y - Y_o) + a_{33}(Z - Z_o)}$$

$$y - y_o = C\frac{a_{21}(X - X_o) + a_{22}(Y - Y_o) + a_{23}(Z - Z_o)}{a_{31}(X - X_o) + a_{32}(Y - Y_o) + a_{33}(Z - Z_o)} \tag{1}$$

These equations are dependent upon the coordinate systems chosen as they involve only coordinates and not distances. It is well known that the distance between two points is invariant to the transformation because the distance between two points is independent of the coordinate system. It is the objective of this paper to derive an invariant form of the projective equations which will involve only the distances, i.e., photo to ground and slant distances (see Figure 1). Thus, the basic equations will not depend upon the type of corrdinate system used. Later it will be shown how such an invariant form of equations will help obtain the rectified ground distance from photo distances measured on single images, and also give us another mathematical model for digital rectification.

MATHEMATICAL FORMULATION

Let us consider two points, 1 and 2, with photo coordinates x_1, y_1 and x_2, y_2; corresponding ground coordinates X_1, Y_1, Z_1 and X_2, Y_2, Z_2; exposure station coordinates X_o, Y_o, Z_o; and principle point coordinates x_o, y_o. To start with, let us use the following notation:

$$x_2 = x_1 + \Delta x \qquad X_2 = X_1 + \Delta X$$
$$y_2 = y_1 + \Delta y \qquad Y_2 = Y_1 + \Delta Y$$
$$Z_2 = Z_1 + \Delta Z$$

$$\text{and } \Delta D = (\Delta X^2 + \Delta Y^2 + \Delta Z^2)^{1/2} \tag{2}$$

The basic projective equations involving photo and ground coordinate data for point 1 are

$$x_1 - x_o = \lambda[a_{11}(X_1 - X_o) + a_{12}(Y_1 - Y_o) + a_{13}(Z_1 - Z_o)] \tag{3}$$

$$y_1 - y_o = \lambda[a_{21}(X_1 - X_o) + a_{22}(Y_1 - Y_o) + a_{23}(Z_1 - Z_o)] \tag{4}$$

$$C = \lambda[a_{31}(X_1 - X_o) + a_{32}(Y_1 - Y_o) + a_{33}(Z_1 - Z_o)] \tag{5}$$

where λ is a scale factor and C is the principal distance.

Similarly, for point 2, with photo coordinates $x_1 + \Delta x$ and $y_1 + \Delta y$,

$$x_1 + \Delta x - x_o = (\lambda + \Delta\lambda)\,[a_{11}(X_1 + \Delta X - X_o) + a_{12}(Y_1 + \Delta Y - Y_o) + a_{13}(Z_1 + \Delta Z - Z_o)] \tag{6}$$

$$y_1 + \Delta y - y_o = (\lambda + \Delta\lambda)\,[a_{21}(X_1 + \Delta X - X_o) + a_{22}(Y_1 + \Delta Y - Y_o) + a_{23}(Z_1 + \Delta Z - Z_o)] \tag{7}$$

$$C = (\lambda + \Delta\lambda)\,[a_{31}(X_1 + \Delta X - X_o) + a_{32}(Y_1 + \Delta Y - Y_o) + a_{33}(Z_1 + \Delta Z - Z_o)] \tag{8}$$

Subtracting Equation 3 from Equation 6 results in

$$\Delta x = \lambda[(a_{11}\Delta X + a_{12}\Delta Y + a_{13}\Delta Z)] + \Delta\lambda[(a_{11}(X_1 + \Delta X - X_o) + a_{12}(Y_1 + \Delta Y - Y_o) + a_{13}(Z_1 + \Delta Z - Z_o)] \tag{9}$$

Similarly,

$$\Delta y = \lambda\,[a_{21}\Delta X + a_{22}\Delta Y + a_{23}\Delta Z] + \Delta\lambda[a_{21}(X_1 + \Delta X - X_o) + a_{22}(Y_1 + \Delta Y - Y_o) + a_{23}\,(Z_1 + \Delta Z - Z_o)], \tag{10}$$

$$0 = \lambda\,(a_{31}\Delta X + a_{32}\Delta Y + a_{33}\Delta Z) + \Delta\lambda[(a_{31}(X_1 + \Delta X - X_o) + a_{32}(Y_1 + \Delta Y - Y_o) + a_{33}(Z_1 + \Delta Z - Z_o)]. \tag{11}$$

PHOTOGRAMMETRIC ENGINEERING AND REMOTE SENSING,
Vol. 53, No. 9, September 1987, pp. 1239–1243

0099-1112/87/5309–1239$02.25/0

Rewriting Equations 9, 10, and 11 and using Equations 3, 4, and 5, we have

$$\Delta x = (\lambda+\Delta\lambda)\,(a_{11}\Delta X+a_{12}\Delta Y+a_{13}\Delta Z) + \frac{\Delta\lambda}{\lambda}\,(x_1-x_o) \tag{12}$$

$$\Delta y = (\lambda+\Delta\lambda)\,(a_{21}\Delta X+a_{22}\Delta Y+a_{23}\Delta Z) + \frac{\Delta\lambda}{\lambda}\,(y_1-y_o) \tag{13}$$

$$0 = (\lambda+\Delta\lambda)\,(a_{31}\Delta X+a_{32}\Delta Y+a_{33}\Delta Z) + \frac{\Delta\lambda}{\lambda}\,c \tag{14}$$

Rearranging Equations 12 through 14, we have

$$\Delta x - \frac{\Delta\lambda}{\lambda}\,(x_1-x_o) = (\lambda+\Delta\lambda)\,(a_{11}\Delta X+a_{12}\Delta Y+a_{13}\Delta Z) \tag{15}$$

$$\Delta y - \frac{\Delta\lambda}{\lambda}\,(y_1-y_o) = (\lambda+\Delta\lambda)\,(a_{21}\Delta X+a_{22}\Delta Y+a_{23}\Delta Z) \tag{16}$$

$$-\frac{\Delta\lambda}{\lambda}\,C = (\lambda+\Delta\lambda)\,(a_{31}\Delta X+a_{32}\Delta Y+a_{33}\Delta Z) \tag{17}$$

Written in matrix form, Equations 15, 16, and 17 become

$$\begin{bmatrix} x - \frac{\Delta\lambda}{\lambda}\,(x_1-x_o) \\ y - \frac{\Delta\lambda}{\lambda}\,(y_1-y_o) \\ -\frac{\Delta\lambda}{\lambda}\,C \end{bmatrix} = (\lambda+\Delta\lambda) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}$$

Because matrix $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$

$$\begin{bmatrix} x - \frac{\Delta\lambda}{\lambda}\,(x_1-x_o) \\ y - \frac{\Delta\lambda}{\lambda}\,(y_1-y_o) \\ -\frac{\Delta\lambda}{\lambda}\,C \end{bmatrix}^{\mathrm{T}} = (\lambda+\Delta\lambda) \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{\mathrm{T}}$$

Define variables as follows:

$$\Delta x^2 + \Delta y^2 = \Delta d^2$$

$$(x_1-x_o)^2 + (y_1-y_o)^2 + C^2 = s_1^2.$$

Define matrix **M** as the orthogonal matrix

$$\mathbf{M} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and $\mathbf{M}(\mathbf{M})^{\mathrm{T}}=\mathbf{I}$, so that we have, after multiplying (**M** being an orthogonal matrix),

$$[\Delta x - \frac{\Delta\lambda}{\lambda}(x_1-x_o)]^2 + [\Delta y - \frac{\Delta\lambda}{\lambda}\,(y_1-y_o)]^2 + (-\frac{\Delta\lambda}{\lambda}\,f)^2 = (\lambda+\Delta\lambda)^2\,\Delta D^2\cdot\mathbf{M}(\mathbf{M})^{\mathrm{T}} = (\lambda+\Delta\lambda)^2\Delta D^2 \tag{18}$$

$$\Delta x^2 + \Delta y^2 + (\frac{\Delta\lambda}{\lambda})^2\,[(x_1-x_o)^2 + (y_1-y_o)^2 + C^2] - 2\,\frac{\Delta\lambda}{\lambda}\,[(x_1-x_o)\Delta x + (y_1-y_o)\Delta y] = (\lambda+\Delta\lambda)^2\Delta D^2 \tag{19}$$

Define the variables

$$p = \frac{\Delta\lambda}{\lambda}$$

$$x_1-x_o = x_1'$$

$$y_1-y_o = y_1'$$

Using these variables,

$$\Delta d^2+p^2s_1^2 - 2p(x_1'\Delta x + y_1'\,\Delta y) = (\lambda+\Delta\lambda)^2\Delta D^2 \tag{20}$$

In terms of direction cosines (see Figure 2) we have

$$\cos \theta_1 = -\frac{(x'_1 \Delta x + y'_1 \Delta y)}{s_1 \, \Delta \, d}.$$

Hence, Equation 20 reduces to

$$\Delta d^2 + p^2 s_1^2 + 2ps_1 \cos \theta_1 \Delta d = \lambda^2 (1+p)^2 \, \Delta D^2. \tag{21}$$

The above equation is to be combined with the following:
Because the scale factor ,λ, is given by (see Figure 2)

$$\lambda = \frac{s}{S} \tag{22}$$

differentiation of Equation 22 leads to

$$p = \frac{\Delta\lambda}{\lambda} = \frac{\Delta s}{s} - \frac{\Delta S}{S}. \tag{23}$$

$$\text{Also, } \Delta D^2 = S_1^2 + (S_1 + \Delta S)^2 - 2S_1(S_1 + \Delta S) \cos \theta \tag{24}$$

This, on simplification, leads to

$$\Delta D^2 = S_1^2 \left[\left(1 + \frac{\Delta S}{S_1}\right) 4 \sin^2 \left(\frac{\theta}{2}\right) + \left(\frac{\Delta S}{S_1}\right)^2 \right]. \tag{25}$$

Equations 20 through 25 are the basic invariant form of the projective equations which contain two distances and enable rectified distances to be obtained.

A comparison of Equation 1 with Equation 21 shows that, whereas only coordinates are involved in Equation 1, distances (photo and ground) are contained in Equation 21. As such, any coordinate transformation will not change the form of Equation 21. The measurement of photo distances will solve the photogrammetric problem. One such example is illustrated in the subsequent paragraph.

It may be noted that

- The Invariant equations involve only distances and, thus, are independent of the coordinate system.
- They are non-linear and, hence, need to be solved by iteration.
- They may be solved for exposure station coordinates if ground distances are known.
- Rectified ground distances can be computed if information about the exposure station coordinates is given.

COMPUTATIONAL PROCEDURE

Let us consider the case where ground distances are to be computed from given photo and exposure station coordinates. It is assumed that information about x_o, y_o, and C is available. To start with we can take

$$\Delta D \approx \frac{\Delta d}{\lambda}. \tag{26}$$

Equation 21 will then give

$$p = \frac{\Delta\lambda}{\lambda} \approx -\frac{2\Delta d \, s_1 \cos \theta_1 + \Delta d^2}{s_1^2 - \Delta \, d^2}. \tag{27}$$

Using this value of p, ΔS can be solved for, which results in the value of ΔD being obtained from the equation. The process can then be repeated.

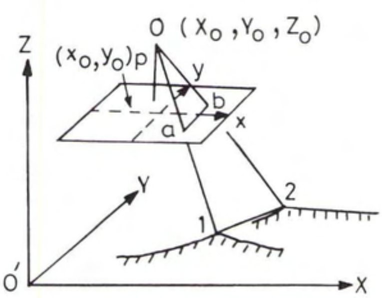

FIG. 1. Geometry of the space resection.

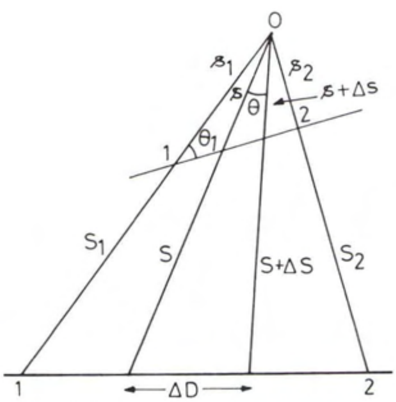

FIG. 2. Photo and ground distances.

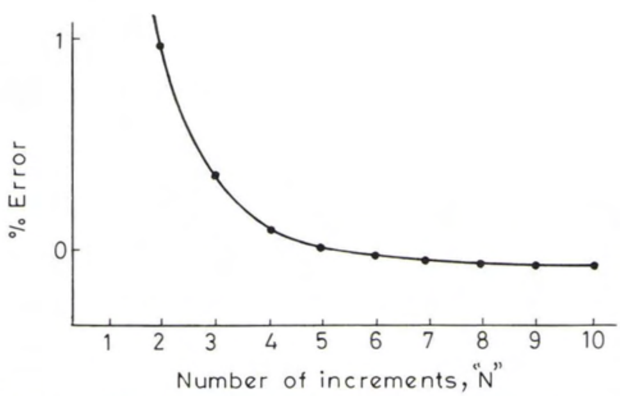

FIG. 3. Percent error versus number of digital increments, "*N*."

NUMERICAL EXAMPLE:

The working out of ground distance is illustrated by taking aerial and ground data from the Casa Grande Test Range, Arizona, published by the Defense Mapping Agency Hydrographic/Topographic Center, Washington, D.C. The data are given in Table 1.

The photo coordinates were measured by the author during his stay at The Ohio State University (Department of Geodetic Sciences) using a Zeiss-PSK Stereocomparator. The exposure station coordinates (X_o, Y_o, Z_o) were obtained by using the RESEC-1 program of the Department of Geodetic Science. The results were

$$X_o = 432585.1824 \pm 2.3949 \text{ m}$$

$$Y_o = 3633272.8913 \pm 0.9075 \text{ m}$$

$$Z_o = 5140.1709 \pm 0.6335 \text{ m}$$

The distance between points 1 and 2, calculated from ground coordinates, was

$$\Delta D = 1611.9354 \text{ metres.}$$

The rectified (computed) distances from photo distances were obtained as follows:

- Compute the photo distance d and the slant distances s_1 and S_1 from the given data.
- Divide the photo distance d between the points 1 and 2 into small segments of equal parts δd by a number N. The selection of 'N' is explained in a later paragraph.
- Solve for p, S, and D using Equations 26, 27, and 22 through 25.
- Repeat the above steps to get the ground distance $D = \Sigma\delta d$. The above procedure was repeated with a different value of N, and the results are shown in Table 2.

Figure 3 shows a plot of number of increments 'N' and the percentage of error. It can be seen that, with $N=6$, the error reaches a minimum. In other words, the photo distance 'd' between two points 1 and 2, with six equal segments of $\delta d = d/N'$ gives good agreement between the given ground distance and the computed distances when using the invariant form of the projective equations.

Another example of the use of the invariant form of the projective equations is the photogrammetric determination of the volume of a symmetrical body. The case illustrated here is the Main Water Tank at California State University, Fresno (See Figure 4).

The camera used was non-metric with a focal length of 75 mm. A single exposure was made from a ground station whose coordinates were determined from a theodolite traverse. The top and bottom of the tank were also fixed from a ground traverse by using a Kern DKM-2 theodolite to determine the vertical height of the tank. The photo distances were obtained using the mono comparator of the Department of Civil Engineering and Surveying, School of Engineering, California State University Fresno. Using the photo distances, it was possible to compute the ground distance along the curve of symmetry of the Water Tank (see Figure 5). Thus, knowing the vertical height (Δh) and the radius (see Figure 6), it was possible to compute the volume of the tank from the simple formula,

$$V = \pi \cdot \Delta h \, \Sigma \left(\frac{r_1 + r_2}{2}\right)^2$$

TABLE 1. GROUND AND PHOTO DATA-CASA GRANDE TEST RANGE (CAMERA FOCAL LENGTH = 152.01 MM) PHOTO NO. 80

Point	Point Identification	Ground Data in Metres			Photo Data in mm	
		X	Y	Z	x	y
1	AE-46	430,823.492	3,634,795.016	432.036	−53,5492	50.0729
2	AF-46	432,435.126	3,634,763.853	435.731	− 1.8000	49.9025
3	AF-45	432,447.333	3,636,323.557	432.940	− 1.8029	100.7271
4	AE-47	430,771.704	3,633,046.953	433.768	−54.5791	−6.0726

TABLE 2. RESULTS OF GROUND DISTANCE COMPUTATIONS

N	$D = \Sigma\delta D$ from photo coordinates	D from ground coordinates	Discrepancy
1	1681.6860 m	1611.9354 m	69.75 m
2	1627.4902 m	"	15.55 m
3	1617.4966 m	"	5.56 m
4	1614.0959 m	"	2.16 m
5	1612.5788 m	"	0.64 m
6	1611.7885 m	"	−0.15 m
7	1611.3332 m	"	−0.61 m
8	1611.0520 m	"	−0.89 m
9	1610.8692 m	"	−1.07 m
10	1610.7455 m	"	−1.19 m

FIG. 4. California State University, Fresno, water tank.

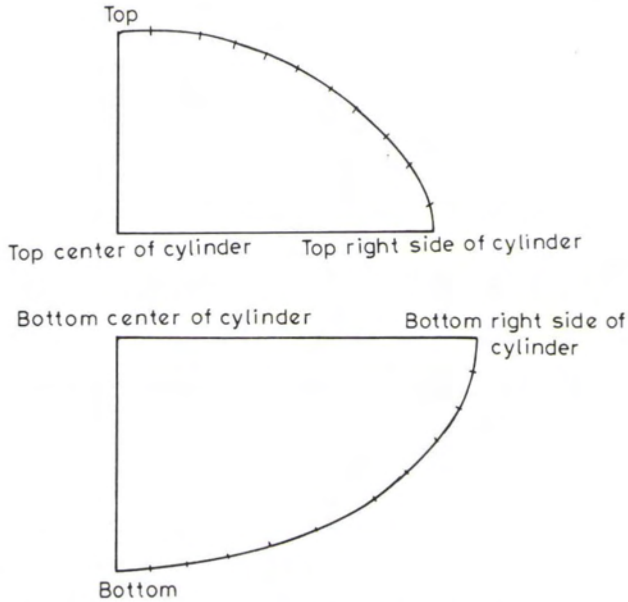

FIG. 5. Vertical cross section of the CSUF water tank.

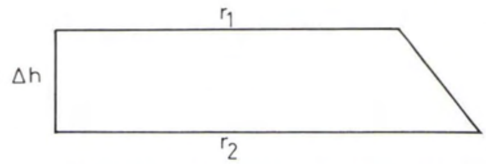

FIG. 6. Horizontal cross section of the CSUF water tank.

The photogrammetric determination gave a value of 157,690.42 gallons whereas the estimated value (from Water Division) CSUF was 150,000 gallons, giving errors of about 5 percent. A more precise value of 'C' could have contributed towards better accuracy. In working out the volume, a perfect symmetry of the tank about the central vertical axis was assumed. This, however, was not the case for actual shape of the tank.

CONCLUSIONS

The new form of the projective equations, derived and proposed here, are non-linear and requires iterations. The advantages of using this form may be stated as follows:

- A single image, as obtained in terrestrial photogrammetry, is enough to yield ground distances. However, the use of more than one should yield better results.
- The equations are valid both for aerial and terrestrial photogrammetry, as shown by the two examples worked out above.

ACKNOWLEDGMENTS

The author acknowledges the work done by Christopher R. Russel and Gary Steven Lewis, Senior Project Students at the Department of Civil Engineering and Survey, CSUF, for the field and computer work done as part of their thesis.

REFERENCES

Rampal, Kunwar K., 1979. A Closed Solution for Space Resection, *Photogrammetric Engineering and Remote Sensing*, Vol. 45, No. 9, pp. 1255–1261.

Slama, C. C (ed.), *Mannual of Photogrammetry*, Fourth Edition, American Society for Photogrammetry and Remote Sensing, Falls Church, Virginia.

(Received 5 September 1986; revised and accepted 7 May 1987)