Optimal Estimation of Displacements by Combining Photogrammetric and Dynamic Models

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ABSTRACT: The dynamic characteristics of displaced object points are integrated with the photogrammetric observation model. The developed formulae, recursively updated for the current state information, are based on the principles of the sequential weighted least-squares adjustment with time consideration. They constitute a formulation of the iterated extended Bayes filter. Test results showed that this approach can improve the final position and accuracy information.

INTRODUCTION

MOST DEFORMATION MODELS are of a static type aiming for a statistical statement showing the existence or non-existence of movements in the space domain. Recent developments focus on dynamic models (Gerstenecker *et al.*, 1978; Papo, 1985) where displacements are studied with respect to time and frequency of occurrence and also as functions of causative parameters (Welsch, 1981).

Photogrammetry has been successfully utilized for monitoring purposes. Subcentimetre accuracies with aerial photogrammetry (Fraser and Gruendig, 1985) and submillimetre accuracies with close-range photogrammetry (Fraser and Brown, 1986) are achievable. However, monitoring photogrammetry has rarely used the time factor extensively as a fourth dimension. Thus, monitoring of displacements is carried out in a static mode (Figure 1). The position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+1)$ are determined solely from the individual photogrammetric campaigns without any interrelation between the two observation epochs (t) and $(t+1)$.

In a time varying situation, however, the object points change their position progressively as a reaction to a cause. In order to take into account the dynamic characteristics of the displacements, we must therefore consider that the parameter vector changes not only as new observations become available, but also as a function of time.

This dynamic information can be directly incorporated into the photogrammetric evaluation model, where monitoring is now performed in a sequential mode (Figure 2). If the functional relationship between the points at successive epochs is adequately known, it is possible to compute a preliminary estimate of the position vector $\mathbf{r}(t)$ based on its previous spatial position $\mathbf{r}(t-1)$. Also, the variance-covariance matrix of $\mathbf{r}(t)$ can be estimated due to the uncertainties of $\mathbf{r}(t-1)$ and of the functional parameters expressing the transition from state $\mathbf{r}(t-1)$ to state $\mathbf{r}(t)$.

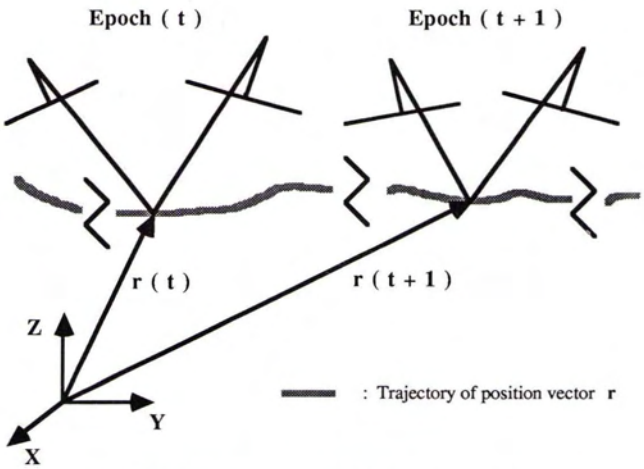

FIG. 1. Determination of displacements in a static mode.

INTEGRATION OF PHOTOGRAMMETRIC AND DYNAMIC MODELS

The sequential mode for monitoring displacements is developed according to the following scenario, which describes a physical situation at an instant of time. This instant of time is characterized by two models:

(a) The *prediction model*, which is defined by two equations.

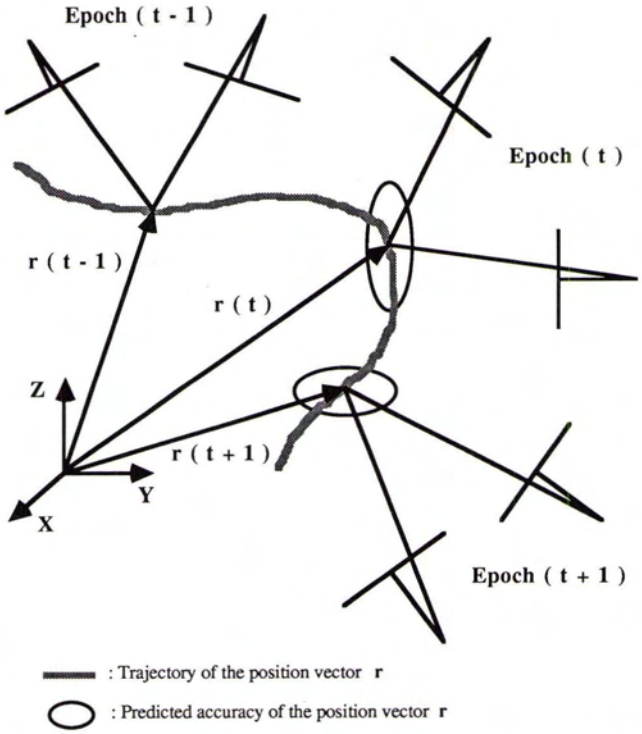

FIG. 2. Determination of displacements in a sequential mode.

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The first equation determines the propagation of the parameter vector through time, that is,

$$\mathbf{x}_{(t)} = \mathbf{T}(t, t-1)\, \mathbf{x}_{(t-1)} + \mathbf{z}(t, t-1) \tag{1}$$

where

- $\mathbf{x}_{(t)}$ is the predicted parameter vector at time t (current state);
- $\mathbf{x}_{(t-1)}$ is the estimated parameter vector at time $t-1$ (previous state);
- $\mathbf{T}(t, t-1)$ is the transition matrix, which maps $\mathbf{x}_{(t-1)}$ onto $\mathbf{x}_{(t)}$ in the time interval $\delta t = (t) - (t-1)$; and
- $\mathbf{z}(t, t-1)$ is the vector describing the uncertainty (residual or noise) of the system.

The second equation determines the propagation of errors of the parameter vector through time, i.e.,

$$\mathbf{C}_{x(t)} = \mathbf{T}(\delta t)\, \mathbf{C}_{\hat{x}(t-1)}\, \mathbf{T}(\delta t)^{\mathrm{T}} + \mathbf{Q}_t \tag{2}$$

where

- $\mathbf{C}_{x(t)}$ is the predicted covariance matrix of $\mathbf{x}_{(t)}$,
- $\mathbf{C}_{x(t-1)}$ is the covariance matrix of the previous state parameter $\mathbf{x}_{(t-1)}$, and
- $\mathbf{Q}_{(t)}$ is the covariance matrix of $\mathbf{z}(t, t-1)$ and generally expresses the uncertainty of the prediction model.

(b) The *photogrammetric model*, which in our case consists of three types of observation equations together with the relevant weight matrices.

(i) The photo-coordinate measurements are related to the unknown parameters through the extended collinearity equations, in a photo-variant mode. The general form of these equations is

$$\mathrm{F}_1\,(\mathbf{x}_I, \mathbf{x}_E, \mathbf{x}_0) = \boldsymbol{\ell}_p;\ \mathbf{P}_P \tag{3}$$

where

$\mathbf{x}_I$ is the vector of the unknown parameters of interior orientation,
$\mathbf{x}_E$ is the vector of the unknown parameters of exterior orientation,
$\mathbf{x}_0$ is the vector of the unknown parameters of object coordinates,
$\boldsymbol{\ell}_p$ is the vector of the observed photo-coordinates, and
$\mathbf{P}_P$ is the weight matrix of the observations.

The Kilpelä-Salmenperä *"physical model"* of additional parameters with $r_0 = 0$ is being used [Kilpelä, 1980].

(ii) The ten elements defining the interior orientation for each photo-frame, namely $x_0, y_0, c, a_1, \ldots, a_7$, are introduced as weighted parameter constraints to avoid ill conditioning. The general form of these equations is

$$\mathrm{F}_2(\mathbf{x}_I) = \boldsymbol{\ell}_I, \text{ with } \boldsymbol{\ell}_I \equiv \mathbf{x}_I^{(0)} \text{ and } \mathbf{P}_I \tag{4}$$

where

$\boldsymbol{\ell}_I$ is the vector of the initially given elements of interior orientation and
$\mathbf{P}_I$ is the weight matrix of the observations.

(iii) The coordinates of the object are also treated as weighted parameter constraints. Although this is necessary for the required control points to avoid datum indeterminancies, all points can be utilized as such. The observation equations have the general form

$$\mathrm{F}_3(\mathbf{x}_O) = \boldsymbol{\ell}_O, \text{ with } \boldsymbol{\ell}_O \equiv \mathbf{x}_O^{(0)};\ \mathbf{P}_O \tag{5}$$

where

$\boldsymbol{\ell}_O$ is the vector of the initially given object coordinates and
$\mathbf{P}_O$ is the weight matrix of the observations.

At a particular instant of time t we have an *a priori* knowledge of the parameters whose *a priori* weight matrix is non-zero (see Equations 1 and 2). Thus, we can write

$$\mathbf{x}_{(t)} = \mathbf{x}_{I(t)} + \delta\mathbf{x}. \tag{6}$$

Omitting the time subscript t and using the subscript 1 denoting the first (or dynamic) model, Equation 6 becomes

$$\mathbf{x}_1 = \mathbf{x}_1 + \delta\mathbf{x}_1 \tag{7a}$$

or

$$\delta\mathbf{x}_1 + \mathbf{x}_1 - \mathbf{x}_1 = \mathbf{O} \tag{7b}$$

or, in a general linearized form,

$$\mathbf{A}_1\delta\mathbf{x}_1 + \mathbf{w}_1 = \mathbf{0},\ (\mathbf{A}_1 = \mathbf{I}) \tag{7c}$$

where

- $\mathbf{A}_1 = \dfrac{\partial f}{\partial \mathbf{x}}$ is the first design matrix,
- $\delta\mathbf{x}_1$ is the solution (correction) vector, and
- $\mathbf{w}_1$ is the misclosure vector.

The weight matrix corresponding to Equation 7c is

$$\mathbf{P}_1 = (\mathbf{C}_{x(t)})^{-1} \text{ or } \mathbf{P}_1 = \mathbf{C}_1^{-1}. \tag{7d}$$

At the same instant of time t the photogrammetric measurement model has the following general linearized form

$$\mathbf{A}_2\delta\hat{\mathbf{x}}_2 - \mathbf{v}_2 + \mathbf{w}_2 = 0 \tag{8a}$$

where

$$\mathbf{A}_2 = \begin{bmatrix} \mathbf{A}_{PI} & \mathbf{A}_{PE} & \mathbf{A}_{PO} \\ 0 & 0 & \mathbf{A}_{OB} \\ \mathbf{A}_{II} & 0 & 0 \end{bmatrix}$$

in which

$$\mathbf{A}_{PI} = \frac{\partial \mathrm{F}_1}{\partial \mathbf{x}_I},\ \mathbf{A}_{PE} = \frac{\partial \mathrm{F}_1}{\partial \mathbf{x}_E},\ \mathbf{A}_{PO} = \frac{\partial \mathrm{F}_1}{\partial \mathbf{x}_O},\ \mathbf{A}_{II} = \frac{\partial \mathrm{F}_2}{\partial \mathbf{x}_I},\ \mathbf{A}_{OB} = \frac{\partial \mathrm{F}_3}{\partial \mathbf{x}_O};$$

$$\mathbf{v}_2 = [\mathbf{v}_P^{\mathrm{T}}\ \mathbf{v}_O^{\mathrm{T}}\ \mathbf{v}_1^{\mathrm{T}}]^{\mathrm{T}}$$

in which $\mathbf{v}_P, \mathbf{v}_I, \mathbf{v}_O$ are the residual vectors corresponding to observation equations $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3$, respectively; and

$$\mathbf{w}_2 = [\mathbf{w}_P^{\mathrm{T}}\ \mathbf{w}_O^{\mathrm{T}}\ \mathbf{w}_1^{\mathrm{T}}]^{\mathrm{T}}$$

in which $\mathbf{w}_P, \mathbf{w}_I, \mathbf{w}_O$ are the misclosure vectors corresponding to the observation equations $\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3$, respectively.

The weight matrix corresponding to Equation 8a is

$$\mathbf{P}_2 = \begin{bmatrix} \mathbf{P}_P & 0 & 0 \\ 0 & \mathbf{P}_O & 0 \\ 0 & 0 & \mathbf{P}_I \end{bmatrix} \text{ or } \mathbf{P}_2 = \mathbf{C}_2^{-1} \tag{8b}$$

The combined linearized mathematical model based on Equations 7c and 8a is

$$\mathbf{A}\delta\hat{\mathbf{x}} - \mathbf{v} + \mathbf{w} = 0 \tag{9a}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{A}_2 \end{bmatrix},\ \mathbf{v} = \begin{bmatrix} 0 \\ \mathbf{v}_2 \end{bmatrix},\ \mathbf{w} = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_1 \end{bmatrix}. \tag{9b}$$

Assuming logically that there is no correlation between the two

sets of measurements, the combined weight matrix for the model of Equation 9a is

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & 0 \\ 0 & \mathbf{P}_2 \end{bmatrix} \tag{9c}$$

The solution vector $\delta\mathbf{x}$ is not partitioned because both subsets of observations are related to the same unknown parameters. It can be estimated by applying the least-squares criterion.

$$\mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} = \delta\hat{\mathbf{x}}_1^{\mathrm{T}}\,\mathbf{P}_1\delta\hat{\mathbf{x}}_1 + \mathbf{v}_2^{\mathrm{T}}\,\mathbf{P}_2\mathbf{v}_2 = \min. \tag{10}$$

In reality, measurements become available sequentially, and/or *a priori* estimates of the solution vector may be available (e.g., Equation 1). Therefore, it is preferable and practical to determine new estimates based on the new measurements (e.g., Equations 3, 4, and 5) in terms of previous solutions. This is possible by deriving sequential expressions of the least-squares solutions (Wells and Krakiwsky, 1971; Junkins, 1978). Hence,

$$\delta\hat{\mathbf{x}} = -\mathbf{N}_1^{-1}\mathbf{q}_1 - \mathbf{N}_1^{-1}\,\mathbf{A}_2^{\mathrm{T}}\,\mathbf{k}_2 \tag{11a}$$

where

$$\mathbf{k}_2 = (\mathbf{P}_2^{-1} + \mathbf{A}_2\mathbf{N}_1^{-1}\,\mathbf{A}_2^{\mathrm{T}})^{-1}\,(-\mathbf{A}_2\mathbf{N}_1^{-1}\,\mathbf{q}_1 + \mathbf{w}_2) \tag{11b}$$

$$\mathbf{N}_1^{-1} = (\mathbf{A}_1^{\mathrm{T}}\,\mathbf{P}_1\mathbf{A}_1)^{-1} = \mathbf{P}_1^{-1} \tag{11c}$$

$$\mathbf{q}_1 = \mathbf{A}_1^{\mathrm{T}}\,\mathbf{P}_1\mathbf{w}_1 = \mathbf{P}_1\mathbf{w}_1 \tag{11d}$$

It is known that

$$-\mathbf{N}_1^{-1}\mathbf{q}_1 = \mathbf{w}_1 = \delta\mathbf{x}_1 \tag{12}$$

where $\delta\mathbf{x}_1$ is the solution for the parameter vector when the first component model only (dynamic model) is used. If we set $\delta\mathbf{x}_2 \equiv \delta\mathbf{x}$ which means that the unknown parameters are estimated after considering the second component model (observation model) and using Equations (11b) through (12), then Equation (11a) becomes

$$\delta\mathbf{x}_2 = \delta\mathbf{x}_1 + \Delta\mathbf{x} \tag{13a}$$

where

$$\Delta\mathbf{x} = -\mathbf{C}_1\,\mathbf{A}_2^{\mathrm{T}}\,(\mathbf{C}_2 + \mathbf{A}_2\,\mathbf{C}_1\,\mathbf{A}_2^{\mathrm{T}})^{-1}\,(\mathbf{A}_2\delta\mathbf{x}_1 + \mathbf{w}_2) \tag{13b}$$

Because we assumed non-linear models for the sake of generality, the final solution $\mathbf{x}$ is determined from both the first and the combined model. When the first model is used, then

$$\hat{\mathbf{x}} \triangleq \hat{\mathbf{x}}_1 = \mathbf{x}^{(0)} + \sum_{i=1}^{k} \delta\hat{\mathbf{x}}_{1i} \tag{14}$$

where k is the required number of iterations.
When the combined model is used, then

$$\hat{\mathbf{x}} \triangleq \hat{\mathbf{x}}_2 = \mathbf{x}^{(0)} + \sum_{i=1}^{m} \delta\hat{\mathbf{x}}_{2i} = \mathbf{x}^{(0)} + \sum_{i=1}^{m} (\delta\hat{\mathbf{x}}_{1i} + \Delta\mathbf{x}_i)$$

$$\text{or } \hat{\mathbf{x}} \triangleq \hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_1 + \sum_{i=1}^{m} \Delta\mathbf{x}_i \tag{15}$$

with m being the required number of iterations.
If we substitute the expression $\Delta\mathbf{x}$ from Equation 13b into Equation 15 and linearize each time about the most recent estimate, we obtain an important recursive formula for the ith iteration

$$\mathbf{x}_{2,i} = \mathbf{x}_1 - \mathbf{C}_1\,\mathbf{A}_2^{\mathrm{T}}\,[\mathbf{C}_2 + \mathbf{A}_2\,\mathbf{C}_1\,\mathbf{A}_2^{\mathrm{T}}]^{-1}\,[\mathbf{A}_2\,(\mathbf{x}_{2,i-1} - \mathbf{x}_1) + \mathbf{w}_2] \tag{16}$$

where $\mathbf{x}_{2,0} = \mathbf{x}_1$ and $\mathbf{x}_1 = \mathbf{x}^{(0)}$ (value at which linearization occurs)

This expression states clearly that, when a new set of observations is added for the determination of the parameter vector, the resulting new parameter vector is equal to the parameter vector estimated from all previous observation equations plus a correction term. Applying the law of error propagation to Equation 16, a sequential form of the variance-covariance matrix $\mathbf{C}_x$ of the estimated parameter $\mathbf{x}$ is derived. Thus,

$$\mathbf{C}_x = [\mathbf{I} - \mathbf{C}_1\,\mathbf{A}_2^{\mathrm{T}}\,(\mathbf{C}_2 + \mathbf{A}_2\mathbf{C}_1\mathbf{A}_2^{\mathrm{T}})^{-1}\,\mathbf{A}_2]\mathbf{C}_1 \tag{17}$$

The dynamic model provides a recursive estimation process through time for the unknown vector of parameters. Therefore, in the above sequential expressions, the time is considered when the parameter vector $\mathbf{x}_{2,i}$ changes not only as new observations become available (term $\Delta\mathbf{x}$) but also as function of cause in time (term $\mathbf{x}_1$). In terms of modern optimal estimation theory, this represents a filtering process. An examination of Equations 16 and 17 derived from the sequential weighted least-squares adjustment with time consideration reveals, that they have the same appearance and, therefore, are mathematically equivalent to the expressions given for the iterated extended Kalman filter for non-linear dynamic systems (Gelb, 1974).

Examining also the structure of these equations with respect to existing familiar forms and computational aspects involved, we observe the following for the term $(\mathbf{C}_2 + \mathbf{A}_2\mathbf{C}_1\mathbf{A}_2^{\mathrm{T}})^{-1}$: First, the sequence of matrices involved does not resemble the well known form of the coefficient matrix $\mathbf{A}_2^{\mathrm{T}}\,\mathbf{C}_2^{-1}\mathbf{A}_2$ of the unknown parameters of the least-squares adjustment. Second, the order n of the matrix to be inverted is much larger than the order u used in a regular photogrammetric bundle block adjustment (n, u are the numbers of observations and unknown, respectively).

At this point we invoke a matrix inversion identity given in Henderson and Searle (1981) which has been found also in Mikhail and Helmering (1973) and Kratky (1980), namely:

$$(\mathbf{C}_2 + \mathbf{A}_2\mathbf{C}_1\mathbf{A}_2^{\mathrm{T}})^{-1} = \mathbf{C}_2^{-1}\,[\mathbf{I} - \mathbf{A}_2(\mathbf{C}_1^{-1} + \mathbf{A}_2^{\mathrm{T}}\mathbf{C}_2^{-1}\mathbf{A}_2)^{-1}\,\mathbf{A}_2^{\mathrm{T}}\,\mathbf{C}_2^{-1}] \tag{18}$$

Also, a new notation is adopted to conform with Gelb's (1974) notation that provides a better understanding of the time factor (Schwarz, 1983) and a more explicit distinction between predicted and updated estimates (Figure 3). The subscript t implies the final estimation at time t, which is obtained after applying the contribution of the measurements of the second model. The symbol $(-)$ indicates predicted values based on the dynamic model immediately prior to time t. The symbol $(+)$ indicates updated values due to the contribution of the observations immediately following the time t.

Applying Equation 18 and this notation on Equations 16 and 17, the final updated expressions are derived (Armenakis, 1987). The final updated estimated parameter vector is determined as

$$\mathbf{x}_{t,i}(+) = \mathbf{x}_t(-) - \mathbf{C}_t(-)\,\mathbf{A}_t^{\mathrm{T}}\,\mathbf{C}_t^{-1}\,\mathbf{G}\,[\mathbf{A}_t\,(\mathbf{x}_{t,i-1}(+) - \mathbf{x}_t(-)) + \mathbf{w}_t] \tag{19}$$

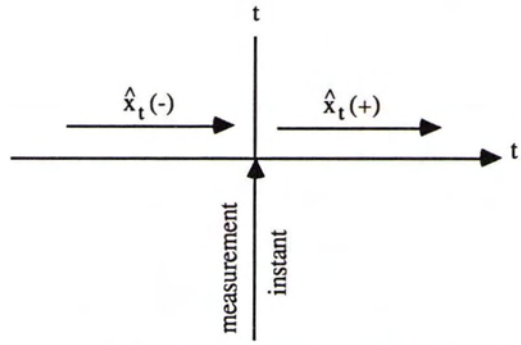

FIG. 3. Predicted and updated values at time t.

and the final updated covariance matrix of the parameter vector is

$$\mathbf{C}_t(+) = [\mathbf{I} - \mathbf{C}_t(-)\,\mathbf{A}_t^{\mathrm{T}}\,\mathbf{C}_t^{-1}\,\mathbf{G}\,\mathbf{A}_t]\,\mathbf{C}_t(-) \tag{20}$$

where

$$\mathbf{G} = \mathbf{I} - \mathbf{A}_t\,(\mathbf{C}_t(-)^{-1} + \mathbf{A}_t^{\mathrm{T}}\,\mathbf{C}_t^{-1}\,\mathbf{A}_t)^{-1}\,\mathbf{A}_t^{\mathrm{T}}\,\mathbf{C}_t^{-1}. \tag{21}$$

These equations represent a formulation of the iterated extended Bayes filter. The use of a different matrix identity (Equation 18) results in a different expression for the so-called Bayes filter (Morrison, 1969; Vanicek and Krakiwsky, 1986).

SEQUENTIAL ESTIMATION OF THE STATE INFORMATION

Practical considerations and computational efficiency led to the use of a reduced measurement model. That is, the object coordinates have been selected to serve as state parameters for the filtering algorithm (Equations 19 and 20). More details about the different situations and approaches examined can be found in Armenakis (1987) and Armenakis and Faig (1987). This solution originates from the fact that we are mainly interested in determining the trajectories of the object points. It involves the following general steps:

Step 1. Solve for the parameters $\mathbf{x}_E$, $\mathbf{x}_I$ of the exterior and interior orientation of the extended space resection.

Step 2. Consider $\mathbf{x}_E$ and $\mathbf{x}_I$ as known and form the reduced photogrammetric model where only the coordinates $\mathbf{x}_O$ of the object points are unknown parameters.

Step 3. Determine the optimal position and accuracy estimates of the current object coordinates using the prediction and reduced observation models in the final updated Equations 19 and 20.

It is understood that the above steps are executed in an iterative manner.

PRACTICAL TEST

The test was conducted in a laboratory environment. The test model (Figure 4) consists of five parts and has the following dimensions 1.40 m by 0.90 m by 0.25 m. In respect to a stable steel frame, the other four parts can accommodate the following deformations:

Part A: rotational area (rotation axis: E-F),
Part B: shift area (parallel to axis E-F),
Part C: stable area, and
Part D: subsidence area (aluminum plate equipped with a loading hook in the center for weights).

In addition, several targetted points can be moved individually.

Two photogrammetric observation epochs were used. In the second epoch single point displacement and subsidence deformation had been introduced in parts of the model. The maximum magnitude of displacements was 1 cm.

Convergent photography with 100 percent overlap was taken from above the four corners of the test field with a Canon AE-1 non-metric camera with standard lens (f=50 mm) and an approximate photo-scale of 1:45. The locations of the object and of the camera stations are illustrated in Figure 5.

The photo-coordinates of the image points on all eight photographs were measured on the precision analog stereo-plotter Wild A-10. Two sets of measurements were performed, resulting in an average accuracy for both x and y photo-coordinates of ± 12 μm. The accuracy of the surveyed object points was ± 1 to ± 2 mm.

The bundle adjustment program PTBV (Photogrammetric Triangulation by Bundles-Photo-Variant; Armenakis, 1987) was utilized to estimate the position and accuracy of the object points in epoch 1. For the second epoch, the predicted object coordinates were estimated through approximate photogrammetric means (without the use of additional parameters) due to lack of a systematic and controlled mechanism causing the displacements. Their uncertainty was introduced by means of the diagonal uncertainty matrix $\mathbf{Q}$ ($\mathbf{q}_{ii}$ = 0.015 m^2).

The sequential estimation of the state information (position and accuracy) was carried out using the integrated photogrammetric and dynamic mathematical model. The computations were performed with the program SPDM (Sequential Photogrammetric Displacement Monitoring; Armenakis, 1987). Besides SPDM, the program PTBV was run as well. The statistical information obtained from both the combined and single epoch bundle block adjustment approaches is given in Table 1.

Finally, the differences in displacements between geodetic results and photogrammetric ones (from SPDM) were compared at 16 points, resulting in average differences of

$$\delta X = -0.4 \text{ mm}, \ \delta Y = 0.1 \text{ mm}, \ \delta Z = -0.9 \text{ mm}.$$

CONCLUSIONS

The comparison and evaluation of the results between the single-epoch bundle adjustment and the combined sequential

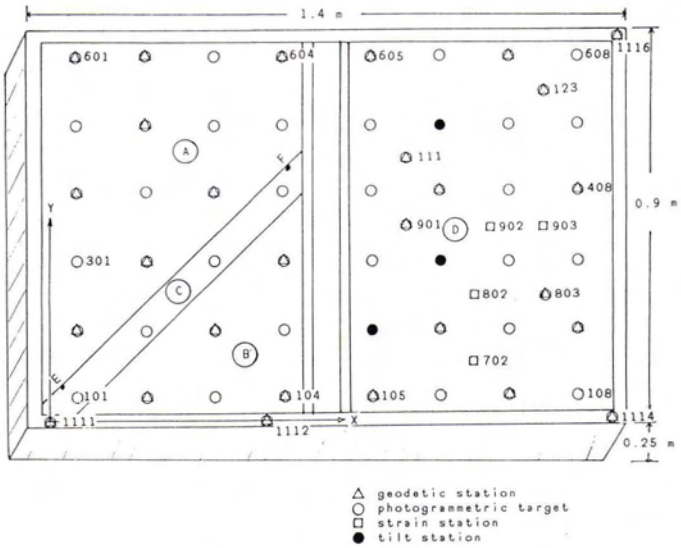

FIG. 4. Layout of the test model.

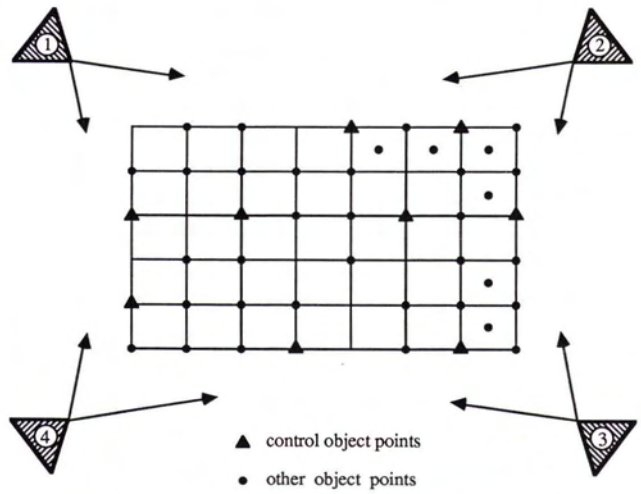

FIG. 5. Plan diagram (not to scale) of the object-camera configuration.

TABLE 1. STATISTICAL INFORMATION OF PHOTOGRAMMETRIC ADJUSTMENTS (EPOCH 2)

Residuals	Mean Value (SPDM)	Mean Value (PTBV)	Standard Deviation (SPDM)	Standard Deviation (PTBV)
photo x (mm)	−0.001	0.000	±0.010	±0.007
photo y (mm)	−0.002	0.000	±0.012	±0.010
check-points X (m)	0.000	0.000	±0.002	±0.002
check-points Y (m)	0.001	0.001	±0.002	±0.002
check-points Z (m)	0.002	0.002	±0.009	±0.009
control-points X (m)	0.000	0.000	±0.001	±0.001
control-points Y (m)	0.000	0.000	±0.001	±0.001
control-points Z (m)	0.000	0.000	±0.003	±0.004

a-posteriori variance factor

(SPDM): 0.181E-09 (PTBV): 0.191E-09

mean value of the standard deviations of the non-control points

(SPDM): $\bar{\sigma}_X = \pm 0.3$ mm $\bar{\sigma}_Y = \pm 0.3$ mm $\bar{\sigma}_Z = \pm 0.4$ mm
(PTBV): $\bar{\sigma}_X = \pm 0.8$ mm $\bar{\sigma}_Y = \pm 0.8$ mm $\bar{\sigma}_Z = \pm 1.2$ mm

photogrammetric approach, based on this experiment and a number of others described in detail in Armenakis (1987), led to the following conclusions:

- When a strong and well-controlled bundle geometry exists, the estimated positional parameters (orientation elements of the exposure stations as well as object coordinates) tend to be quite similar. This is illustrated in Table 1 where the mean values and the standard deviations of the examined residuals and the *a-posteriori* variance factors do not significantly differ. The imposition of additional object constraints is reflected in the slightly larger standard deviations of the photo-residuals as well as in slight differences in the estimated elements of interior orientation.
- The estimated positional parameters show differences for the two approaches in cases of weak bundle geometry and/or poor datum definition. The combined sequential photogrammetric approach always provides better absolute results when the predicted information (object coordinates and their accuracies) is reliable.
- The use of the combined sequential photogrammetric approach provides a significant improvement to the absolute accuracy of the object coordinates. This is shown in Table 1 where the mean values of the standard deviations of the non-control points are smaller than their corresponding values from the single-epoch bundle adjustment approach.

Generally, it can be stated that the integration of photogrammetric and dynamic information contributes to a better estimation of position and accuracy, because each model plays the role of a safeguard and complements the other. The recursive nature of the approach has great potential in real-time photogrammetric applications.

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