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Two Non-Iterative Methods of Determining the Orientation of a Stellar Camera

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ABSTRACT: Stellar photography has found use in the calibration of mapping cameras and in determining the orientation of the stellar cameras serving as attitude sensors, e.g., the lunar mapping cameras on the NASA Apollo 15, 16, and 17 missions. The equations used to relate the image measurements to the attitude of the camera are the collinearity equations of photogrammetry. These require initial approximations to the unknown attitude angles and the use of an iterative Gaussian correction algorithm. Two simpler methods are described which do not require initial approximations to the unknown angles. One of the methods is a modification of the Church method of space resection and the second is based on the mathematical formulation of a rotation transformation about a single axis. The methods require only two images for a solution. Modifications are given for using the methods with aerial and terrestrial photographs when the camera position is known and two or more control points are imaged on a photograph.

INTRODUCTION

STELLAR PHOTOGRAPHY has found use in photogrammetry in the calibrations of aerial cameras and in determining the orientation of stellar cameras which served as attitude sensors for the lunar mapping camera on the NASA Apollo 15, 16, and 17 missions. The reasons for these two uses are that the positions of the stars are accurately known and stars closely approximate point light sources at infinity. This paper addresses the problem of determining the orientation of the camera itself, but some of the findings may have broader implications. From the photogrammetrist's standpoint, the problem of determing the angular orientation of a stellar camera is a special case of space resection or absolute orientation. It is special in the sense that the position of the camera is known, which eliminates three of the six orientation parameters from the unknowns list. The techniques illustrated for stellar photographs can be applied, with slight modifications, to aerial and terrestrial photographs. For each type of photograph the elements of the problem are the known position coordinates of the taking camera, the known coordinates of two or more control points, and the three unknown orientation angles of the camera.

The equations which have been used to determine the attitude of a stellar camera are the collinearity equations of photogrammetry. While the collinearity equations are mathematically rigorous and yield accurate solutions to a variety of problems, their solution requires initial approximations to the unknown attitude angles and uses a Gaussian differential correction algorithm which typically requires several iterations for convergence. The objective of this paper was to find possible alternatives to the collinearity equations which would not require initial approximations to the orientation angles and which would require only two star images for a solution. Two such methods were derived. One is a modification of the Church resection method and the other uses a single rotation transformation.

MODIFICATION OF THE CHURCH METHOD

The Church method of space resection was developed in 1945 for aerial photographs and is well suited for manual calculation because the largest system of simultaneous equations involves only three equations. The method is described in detail in the *Manual of Photogrammetry* (Slama, 1980). Basically, the Church method is a two-stage process. The first stage consists of an iterative solution for the coordinates of the camera position. However, this first stage is not applicable to stellar photographs because the position of the camera is already known. The second stage of the Church method of resection consists in solving simultaneous linear equations for the nine elements of the orientation matrix. It is this second stage that is applicable to stellar photographs.

As is pointed out in the *Manual of Photogrammetry*, the following relationship holds for one star:

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} \tag{1}$$

where the following definitions apply:

- x_i, y_i = photographic coordinates of the i^{th} star image,
- U_i, V_i, W_i = direction cosines of the i^{th} star in the celestial coordinate system,
- $[\mathbf{M}_s]$ = orientation matrix of the stellar camera in the celestial coordinate system,
- f = stellar camera principal distance (focal length), and
- P_i, Q_i, R_i = direction cosines of the i^{th} star image in the camera coordinate system.

These direction cosines of the camera image ray are given by the equations

$$P_i = \frac{x_i}{(x_i^2 + y_i^2 + f^2)^{1/2}}$$

$$Q_i = \frac{y_i}{(x_i^2 + y_i^2 + f^2)^{1/2}}$$

$$R_i = \frac{-f}{(x_i^2 + y_i^2 + f^2)^{1/2}} \tag{2}$$

For three stars whose images do not lie on the same straight line (insuring that the image vectors are linearly independent),

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the relationships given by Equation 1 can be expressed by a single matrix equation written in the form

$$\begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} = [\mathbf{M}_s] \begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} \tag{3}$$

Equation 3 can be solved for $[\mathbf{M}_s]$ by postmultiplying the leftmost matrix by the inverse of the rightmost matrix so that

$$[\mathbf{M}_s] = \begin{bmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{bmatrix} \begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix}^{-1} \tag{4}$$

Equation 3 may be used to solve for the orientation by using only two stars rather than three. To do this an artifice is used to create a fictitious third star image and a corresponding fictitious third star. The direction cosine vector of the fictitious star image in the camera system is formed by taking the cross product of the direction cosine vectors of two real star images: i.e.,

$$[P_1,Q_1,R_1]' \times [P_2,Q_2,R_2]' = a[P_3,Q_3,R_3]' \tag{5}$$

where a is the sine of the angle between the two camera system vectors. The corresponding celestial system direction cosine vector for the fictitious star is formed by taking the cross product of the direction cosine vectors of the two real stars: i.e.,

$$[U_1,V_1,W_1]' \times [U_2,V_2,W_2]' = b[U_3,V_3,W_3]' \tag{6}$$

where b is the sine of the angle between the two celestial system vectors. $[P_3,Q_3,R_3]'$ and $[U_3,V_3,W_3]'$ are entered into Equation 3.

The advantage of being able to perform a noniterative solution with only two stars may not seem significant at first. However, if all possible solutions using small subsets of the stellar field are considered, it is seen that the number of combinations for 20 stars taken two at a time is 190 while the number of combinations for 20 stars taken three at a time is 1140. Also, there is a timesaving factor involved where maximum accuracy is not required because it is probably faster to create the fictitious third star than to identify a real third star. Finally, this technique can be adapted to aerial and terrestrial photography where control points may be very few.

A SINGLE ROTATION METHOD

Single rotation solutions to three-dimensional rotation transformation problems are seldom, if ever, used in photogrammetry, the preferred approach being to define a rotation transformation in terms of three angles and three axes. In the physical world, the corresponding transformation can be accomplished by a single rotation about a single well chosen axis. As it turns out, the analytical expression of the physical rotation can be employed to determine the attitude of a stellar camera in a non-iterative fashion using only two stars. While the solution is neither brief nor elegant, it does demonstrate how the relationship between the single rotation and the three corresponding orientation angles can be used to solve a practical problem.

To begin, the camera system direction cosines of the star images are computed and the celestial system direction cosines of the stars are computed. It is assumed that the camera axes and the celestial axes are coincident initially. So, in effect, the direction cosine vectors of the stars and their corresponding images represent points on a sphere of unit radius. The object is to find an axis that passes through the center of the sphere and an angle of rotation about the axis that will rotate the points that represent the stars on the unit sphere into the points that represent their images. This will define a rotation transformation which, when performed on the camera axes, will represent the attitude of the camera axes at the time of exposure.

The first task is to solve for the axis of rotation. The axis is defined by the cross product of two vectors as is illustrated in Figure 1. The first vector is one whose initial point is the celestial direction cosine vector of the first star and whose terminus is the vector of camera direction cosines for that star. That is,

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} P_1 - U_1 \\ Q_1 - V_1 \\ R_1 - W_1 \end{bmatrix}. \tag{7}$$

A second vector is defined similarly for the second star.

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} P_2 - U_2 \\ Q_2 - V_2 \\ R_2 - W_2 \end{bmatrix} \tag{8}$$

The cross product is then taken to find the rotation axis.

$$\begin{bmatrix} X_T \\ Y_T \\ Z_T \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \times \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \tag{9}$$

For computational convenience, a unit vector $[X_R,Y_R,Z_R]'$ proportional to $[X_T,Y_T,Z_T]'$ is formed: i.e.,

$$\begin{bmatrix} X_R \\ Y_R \\ Z_R \end{bmatrix} = a \begin{bmatrix} X_T \\ Y_T \\ Z_R \end{bmatrix} \tag{10}$$

where

$$a = (X_T^2 + Y_T^2 + Z_T^2)^{-1/2}. \tag{11}$$

It should be noted that, if the two difference vectors $[X_1,Y_1,Z_1]'$

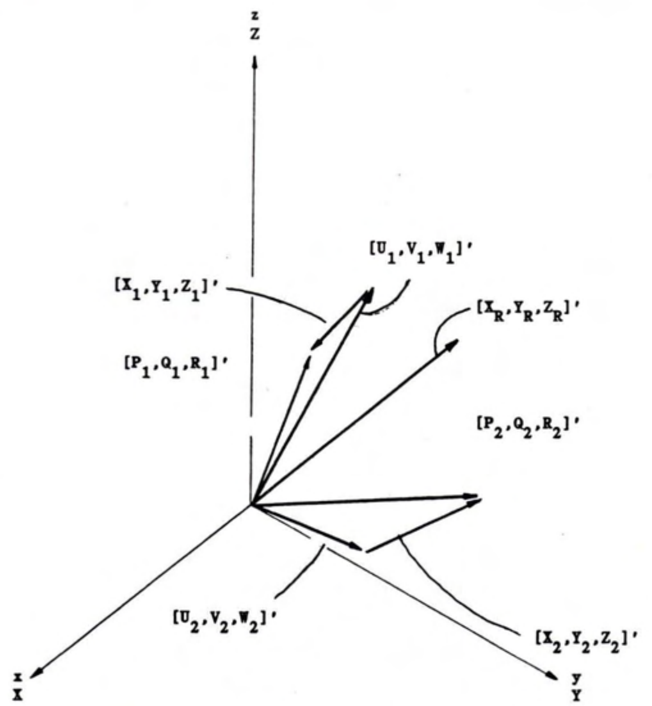

FIG. 1. The cross product of the difference vectors $[X_1, Y_1, Z_1]'$ and $[X_2, Y_2, Z_2]'$ defines the rotation axis $[X_R, Y_R, Z_R]'$.

and $[X_2,Y_2,Z_2]'$ are parallel, they cannot be used to find the rotation axis. Likewise, if they are even only nearly parallel, a solution found by this method will be geometrically weak.

The next step is to solve for the rotation angle ρ. The projection of the celestial direction cosines vector $[U_1,V_1,W_1]'$ onto the rotation axis (illustrated in Figure 2) is the cosine of the angle between the vector and the axis multiplied times a unit vector in the direction of the rotation axis or, in equation form,

$$\begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} = d \begin{bmatrix} X_R \\ Y_R \\ Z_R \end{bmatrix} \tag{12}$$

where

$$d = U_1X_R + V_1Y_R + W_1Z_R. \tag{13}$$

The projection of the camera direction cosines vector $[P_1,Q_1,R_1]'$ onto the rotation axis $[X_R,Y_R,Z_R]'$ is also $[C_X,C_Y,C_Z]'$ because $[U_1,V_1,W_1]'$ is being rotated into $[P_1,Q_1,R_1]'$ about the rotation axis.

The magnitude of the rotation angle ρ is found from the equation

$$\cos \rho = \frac{[U_1-C_X,V_1-C_Y,W_1-C_Z]' \cdot [P_1-C_X,Q_1-C_Y,R_1-C_Z]'}{|[U_1-C_X,V_1-C_Y,W_1-C_Z]'|\ |[PO_1-C_X,Q_1-C_Y,R_1-C_Z]'|}. \tag{14}$$

Equation 14 gives the cosine of an angle between 0° and 180°. To ascertain whether this is a positive or negative angle, the cross product $[U_1-C_X,V_1-C_Y,W_1-C_Z]' \times [P_1-C_X,Q_1-C_Y,R_1-C_Z]'$ is formed. If it has the opposite direction of $[C_X,C_Y,C_Z]'$, then the rotation angle is negative.

Now that the rotation axis and the rotation angle have been found, the next step is to determine the point $[A_2,B_2,C_2]'$ which represents an arbitrary point $[A_1,B_1,C_1]'$ after it has been transformed by the rotation. First, $[A_1,B_1,C_1]'$ is projected onto the rotation axis to form the vector $[A_X,B_Y,C_Z]'$. To calculate the components A_X,B_Y, and C_Z it is first necessary to find the cosine of θ, the angle between the position vector $[A_1,B_1,C_1]'$ and the rotation axis $[X_R,Y_R,Z_R]'$ from the equation

$$\cos \theta = \frac{[A_1,B_1,C_1]' \cdot [X_R,Y_R,Z_R]'}{|[A_1,B_1,C_1]'|\ |[X_R,Y_R,Z_R]'|} \tag{15}$$

and the components of the projection are given by

$$A_X = (X_R)(\cos \theta)(A_1^2 + B_1^2 + C_1^2)^{-1/2}$$

$$B_Y = (Y_R)(\cos \theta)(A_1^2 + B_1^2 + C_1^2)^{-1/2}$$

$$C_Z = (Z_R)(\cos \theta)(A_1^2 + B_1^2 + C_1^2)^{-1/2} \tag{16}$$

Next, the two vectors from $[A_X,B_Y,C_Z]'$ to the given point $[A_1,B_1,C_1]'$ and to the unknown point $[A_2,B_2,C_2]'$ are formed. They are defined as follows:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} - \begin{bmatrix} A_X \\ B_Y \\ C_Z \end{bmatrix} \tag{17}$$

and

$$\begin{bmatrix} D \\ E \\ F \end{bmatrix} = \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} - \begin{bmatrix} A_X \\ B_Y \\ C_Z \end{bmatrix} \tag{19}$$

The magnitudes of these two vectors are equal so that $A^2 + B^2 + C^2$ equals magnitude of $[A,B,C]'$ times the magnitude of $[D,E,F]'$. Then

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} \cdot \begin{bmatrix} D \\ E \\ F \end{bmatrix} = (A^2 + B^2 + C^2) \cos \rho \tag{20}$$

or

$$AD + BE + CF = (A^2 + B^2 + C^2) \cos \rho. \tag{21}$$

Also

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} \times \begin{bmatrix} D \\ E \\ F \end{bmatrix} = (A^2 + B^2 + C^2) \sin \rho \begin{bmatrix} X_R \\ Y_R \\ Z_R \end{bmatrix} \tag{22}$$

from which the following equations are formed:

$$BF - CE = X_R (A^2 + B^2 + C^2) \sin \rho \tag{23}$$

$$CD - AF = Y_R (A^2 + B^2 + C^2) \sin \rho \tag{24}$$

Solving Equations 23 and 24 for D and E and substituting in Equation 20 gives

$$D = \frac{AF + (A^2 + B^2 + C^2)Y_R \sin \rho}{C} \tag{25}$$

$$E = \frac{BF - (A^2 + B^2 + C^2)X_R \sin \rho}{C} \tag{26}$$

$$F = B(X_R \sin \rho) - A(Y_R \sin \rho) + C \cos \rho \tag{27}$$

Substituting D, E, and F into Equation 18 gives

$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} D \\ E \\ F \end{bmatrix} + \begin{bmatrix} A_X \\ B_Y \\ C_Z \end{bmatrix} \tag{28}$$

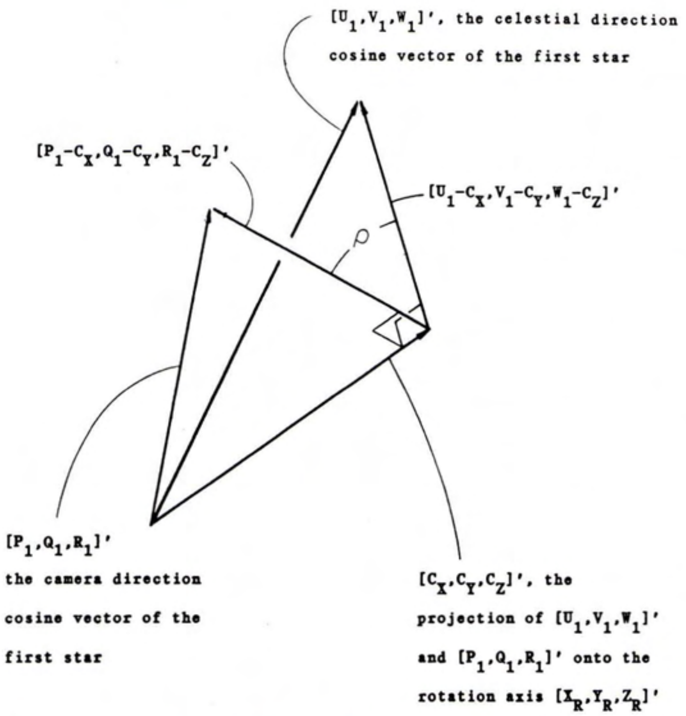

FIG. 2. The angle of rotation ρ is the angle between the vectors $[U_1 - C_X, V_1 - C_Y, W_1 - C_Z]'$ and $[P_1 - C_X, Q_1 - C_Y, R_1 - C_Z]'$.

which is the vector $[A_1, B_1, C_1]'$ after being rotated. The rows of the orientation matrix are then computed by setting the position vector $[A_1, B_1, C_1]'$ successively equal to $[1,0,0]'$, $[0,1,0]'$, and $[0,0,1]'$ and computing the transformed position vectors whose components are the elements of the orientation matrix.

Although there are several steps in this solution, it is fairly straightforward and an interesting application of the single rotation concept. It is included to show that a non-iterative two-star solution is possible without having to resort to fictitious stars as was done in the modification of the Church method for two stars.

MODIFICATIONS FOR AERIAL AND TERRESTRIAL PHOTOGRAMMETRY

The methods described thus far may be applied to aerial and terrestrial photogrammetry for cases in which the camera position is known and in which the rectangular coordinates of two or more imaged control points are known. The major difference that must be accounted for is that the direction cosines U_i, V_i, and W_i of the i^{th} control point in the object space are not defined in terms of right ascension and declination on the celestial sphere, but in terms of rectangular coordinates of some local coordinate system. Typically, if

X_i, Y_i, Z_i = the rectangular coordinates of the i^{th} control point

and

X_C, Y_C, Z_C = the rectangular coordinates of the camera, then

$$U_i = \frac{X_i - X_C}{[(X_i - X_C)^2 + (Y_i - Y_C)^2 + (Z_i - Z_C)^2]^{1/2}}$$

$$V_i = \frac{Y_i - Y_C}{[(X_i - X_C)^2 + (Y_i - Y_C)^2 + (Z_i - Z_C)^2]^{1/2}} \qquad (28)$$

$$W_i = \frac{Z_i - Z_C}{[(X_i - X_C)^2 + (Y_i - Y_C)^2 + (Z_i - Z_C)^2]^{1/2}}$$

These values, rather than the corresponding celestial system values defined by Equations 2, may be entered into the appropriate equations of the previous sections. As aircraft positioning systems are improved, the number of occasions where the coordinates of the camera are accurately known but where its attitude is unknown may increase. In these cases there may be more opportunities to use the methods described in this paper.

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