Dominant Geometry Combinations of Two- and Three-Point Perspective in Close-Range Applications

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ABSTRACT: In every close-range perspective image there is a dominant geometry that controls the photogrammetric analysis and allows the reconstruction of the imaged three-dimensional scene. The analysis of multiple two-point perspective geometry and combined two- and three-point geometry provides additional application tools for close-range imagery. When there are multiple or combined geometries, parameters such as the camera station and principal point are shared, so less information is required for three-dimensional reconstruction than when simple geometry stands alone. This paper discusses application procedures for these multiple and combined geometries, with emphasis on the least-squares solution of multiple lines to a vanishing point, the analytical location of the true horizon line (THL) in a two-point solution, the finding of the camera station, and the conversion of object-space coordinates between two- and three-point perspective solutions.

INTRODUCTION

IN SINGLE-PHOTOGRAPH PERSPECTIVE, dominant geometry provides the best way for the photogrammetric analyst to reconstruct the three-space dimensions of the object imaged. In two previous papers (Williamson and Brill, 1987; Brill and Williamson, 1987) the dominant geometry was either two- or three-point perspective, and was assumed sufficient for a complete reconstruction. (For one-point perspective, see Beamish (1984) and Slama (1980).) Now, we address the application of multiple two-point and combined two- and three-point perspective geometries. A cropped perspective image with multiple dominant geometries can require less prior information than a cropped image with single dominant geometry. Multiple perspective and combined geometries can offer enough redundancy to strengthen a weak solution. These facts motivate using multiple and combined perspective geometries in three-dimensional reconstruction.

One combined geometry is multiple two-point perspective. In two-point imagery, numerous objects can rest on parallel horizontal planes, and have some of the same Phase-1 (image-space) parameters, e.g., vanishing points (**VPs**), principal point (**pp**), camera station (**CSXY**), effective focal length (**f′**), and rotation angles. For example, if the vanishing points for a number of buildings (in one image) lie on the same **THL**, the **CSXY**s are identical. This identity enables retrieval of the Phase-1 geometry required to develop the model-space coordinates, even if the imagery is cropped and no other information is available. Then, given at least one known dimension, the object-space (Phase-2) coordinates can be determined.

Another combined geometry contains two-point and three-point perspective. (The geometry of three-point perspective exists in any two-point perspective image that contains an inclined plane.) When the tilt angle is 90° (and the object parallels the ground plane, but is rotated to the image plane), there is two-point perspective, and vertical lines in object space are truly parallel in image space. However, this applies only to the objects that create the two-point geometry on the image, while other objects in the image can appear as different perspective views. For example, if a Cape Cod house is photographed with a camera tilt angle of 90°, the main walls – and all planes orthogonal to the main walls – produce a dominant two-point geometry. However, the roof (an inclined plane, as illustrated in Figure 1) will be in a three-point perspective that shares the two-point Phase-1 parameters **VPY**, **pp**, and **f′**. The three-point reconstruction can be done without full format or the parametric values of the Phase-1 perspective, but only after first working the two-point reconstruction. In particular the point $\mathbf{VPZ}_3$ (subscripts refer to two- or three-point perspective values) can be found graphically by constructing three perpendiculars: (1) from $\mathbf{THLXY}_3$ through the **pp**, (2) from $\mathbf{pp\text{-}VPX}_3$ through **VPY**, and (3) from **pp-VPY** through $\mathbf{VPX}_3$. The intersection of the three lines is $\mathbf{VPZ}_3$. The analytical procedure is essentially the same, making use of the two **VP**s and the **pp**. The solution to the remaining three-point Phase-1 parameters follows standard application procedures.

In this paper, we discuss analytical and graphical reconstruction applications, with emphasis on the analytical applications. Perspective reconstruction methods require that certain parametric values be either known or determined through the available geometry (Moffitt and Mikhail, 1980); our emphasis will be on the latter case. We also discuss analytical procedures that parallel the graphical methods: conversion between two-point and three-point object-space coordinates for the same image, the least-squares solution of multiple lines to a vanishing point, and the analytical location of the true horizon line in a two-point solution.

The applications discussed are general in development, and can be adapted to similar situations. We recommend that, prior to any graphical or analytical procedures, the photogrammetric analyst should list what is observed and make limited sketches of the prospective procedures, even if the approach is to be analytical. Such preliminary analysis commonly reveals subtle information about solutions, and therefore is a good investment of time.

As in our previous two papers, the methods in this paper derive from standard techniques of architectural drawing (McCartney, 1963; Walters and Bromham, 1970), applied so as to infer three-dimensional geometry from an image. These graphical methods have analytical counterparts that are also used in architecture (Yacoumelos, 1970). The techniques have been discussed, in part, by Gracie *et al.* (1967) and Kelley (1978–1983), and as before we present them here for broader dissemination.

ANALYTICAL SOLUTIONS FOR VANISHING POINTS

The analytical methods discussed in this paper rely on some fundamental techniques to find the vanishing points. These

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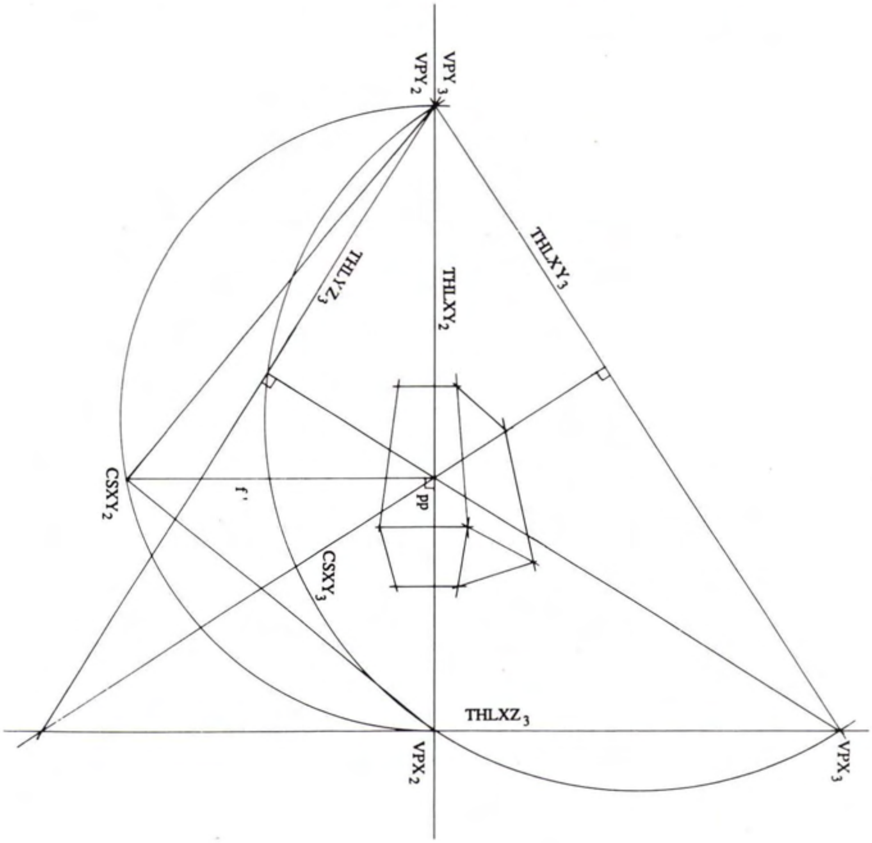

FIG. 1. Two- and three-point perspective in same image — Cape Cod house.

techniques can be used no matter what the perspective in the imagery, e.g., one-, two-, or three-point perspective. For example, the image of a warehouse and loading dock illustrated in Figure 2a appears in two-point perspective. We use parallel lines on the horizontal planes of the warehouse and loading dock to obtain the analytical solution for a vanishing point. In this two-point perspective example there are at least eight parallel lines (vanishing lines) converging to a point (a vanishing point). To determine the vanishing point, one can use equations for two intersecting lines, or for three or more intersecting lines. (It is sometimes advisable to use the two-line intersection procedure for quick approximations of the vanishing-point location. The application equations for determining the intersection of two lines are given in Section I of the Appendix.) Although only two points are needed to determine a line, Section I of the Appendix may not offer the desired line intersection accuracy. More measured points per line, and a least-squares solution (not discussed further here), might provide more accuracy.

For multiple parallel perspective lines ($k = 1, m$), each defined by two points, the equations to be solved for a common intersection point are

$$A_{1k}\, x + A_{2k}\, y = C_k \tag{1}$$

where x, y are the coordinates of a vanishing point. The equations for A_{1k}, A_{2k}, and C_k are

$$A_{1k} = y_{2k} - y_{1k},$$
$$A_{2k} = x_{1k} - x_{2k}, \text{ and} \tag{2}$$
$$C_k = x_{1k}\, y_{2k} - y_{1k}\, x_{2k},$$

where (x_{1k}, y_{1k}) and (x_{2k}, y_{2k}) are two points on the same line k.

In matrix-vector notation, Equation 1 can be written

$$\mathbf{A} \begin{vmatrix} x \\ y \end{vmatrix} = \mathbf{C}. \tag{3}$$

The least-square best fit solution for x, y is obtained by premultiplying both sides by $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$, to obtain the estimate

$$\begin{vmatrix} x_i \\ y_i \end{vmatrix} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{C}. \tag{4}$$

The distance D_k from each line to the point x_i, y_i is given by

$$D_k = [(x_{1k} - x_i + t(x_{2k} - x_{1k}))^2 + (y_{1k} - y_i + t(y_{2k} - y_{1k})^2]^{1/2}, \tag{5}$$

where

$$t = t1/t2, \tag{6}$$

and

$$t1 = (x_{2k} - x_{1k})(x_i - x_{1k}) + (y_{2k} - y_{1k})(y_i - y_{1k})$$
$$t2 = (x_{2k} - x_{1k})^2 + (y_{2k} - y_{1k})^2.$$

A program listing of this procedure is in Section II of the Appendix. The value of D_k is the minimum distance error with which the particular line k participates in the solution for the vanishing point. Hence, to improve precision, lines with large values of D_k can be discarded and the least-squares solution performed again with the remaining lines.

DETERMINING THE TWO-POINT THL

The examples in our previous papers always had enough sets of parallel lines to define the vanishing points, e.g., two sets of

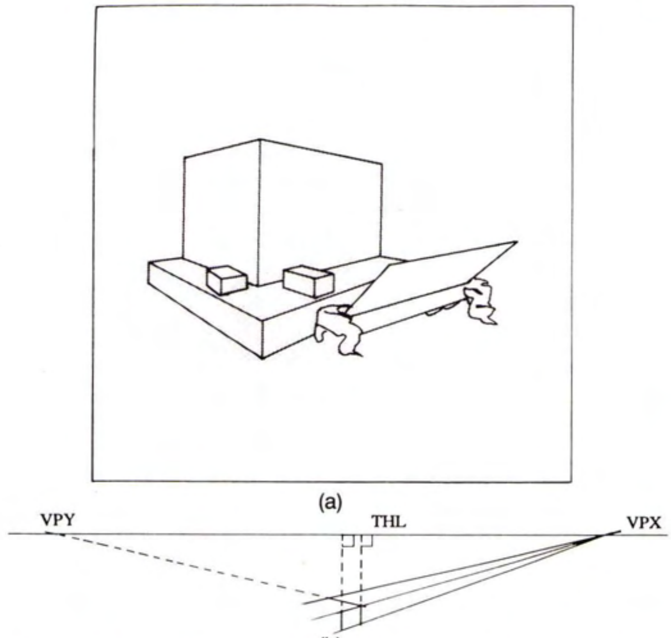

FIG. 2. (a) Two- and three-point perspective in same image — warehouse and trailer. (b) Finding two-point THL in subset of Figure 2a.

lines for a two-point solution. Two vanishing points are needed to define a **THL**, and two sets of converging lines are sufficient to determine these vanishing points. However, this condition can often be absent in real life. For example, consider Figure 2b (a subimage in Figure 2a) that is cropped so only **VPX** can be established directly, and only the top right line of the crate is available to locate **VPY**. To determine the position of the THL, locate **VPX**, and draw a line perpendicular to the imaged vertical lines through **VPX**: this line is the **THL**. Locate VPY by extending the right top line of the crate to the THL. The analytical solution is to use the equation of the perpendicular distance from a line to a given point to locate the THL. **VPY** is then determined using the two-line intersection method, where the two lines are the THL and the line on the top of the crate.

SINGLE IMAGE CONTAINING MULTIPLE TWO-POINT PERSPECTIVE

An example of multiple two-point perspective (see Figure 2a) is an illustrated image of a warehouse and a tilted trailer parked close to the loading dock. The loading dock contains crates with various sizes and positions, all appearing in two-point perspective. Figure 2a, representing a cropped frame, will be used in our multiple two-point perspective discussion.

One of the easiest two-point solutions is when two rectangular objects, resting on parallel horizontal planes, are not parallel to each other. For example, two large crates on the loading dock rest on the same horizontal plane, but the corresponding sides of each crate are not parallel. The dominant geometry is multiple two-point perspective (see Figure 3). Each crate gives a different two-point perspective solution. Except for a known dimension on one of the crates, there is no additional information (such as a known diagonal angle). This is a likely problem of multiple two-point perspective.

The two crates resting on the loading dock have the same **THL**. The multiple two-point perspective of the crates will have other common parameters: **f′**, **pp**, tilt angle (**t**), swing angle (**s**), and—most importantly—the **CSXY**. In this example, to solve for the **CSXY** is the key to the problem. Figure 4 shows the two crates and the axes of a measuring coordinate system. The **THL**

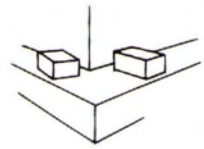

FIG. 3. Cropped image, multiple two-point perspective — two crates on loading dock.

can be located when the positions of the two sets of vanishing points (VPX1, VPY1, VPX2, and VPY2) have been determined. The two sets of vanishing points define the diameters of two circles whose centers (C1 and C2) are on the **THL**. In both perspective geometries, the **CSXY** is defined to lie on the circle with VPX-VPY as a diameter. The position of **CSXY** is common to each of the two-point solutions, and therefore must be an intersecting point of the two circles. Two such intersection points exist, but for convenience we shall select the **CSXY** to be below the **THL**. The point **CSXY** is a vertex of the triangle C1-**CSXY**-C2. Two sides of the triangle are radii R1 and R2 of the circles, and the third side is the distance D12 between the centers of the circles.

A simple analytical procedure to determine this multiple perspective **CSXY** is listed in Sections I to III of the Appendix. Locate the four vanishing points VPX1, VPY1, VPX2, and VPY2 for the two crates, using one of the vanishing-point procedures, and placing the origin of coordinates at VPY1 with x-axis towards VPX2. Compute the coordinates of the centers C1, C2 of the two-point semicircles, and also their radii R1, R2. The centers are easily found by averaging the respective x and y coordinates of the vanishing points, e.g., $Cx = (VPXx + VPYx)/2$, and $Cy = (VPXy + VPYy)/2$. The radii are determined from halving the distance between the same vanishing points, e.g., $R = [(VPXx - VPYx)^2 + (VPXy - VPYy)^2]^{1/2}/2$. The distance between centers (D12) is found in a similar manner. Compute the coordinates of the **CSXY** using the law of cosines on triangle C1-C2-**CSXY**. The graphical solution is shown in Figure 4.

Once the **CSXY** is determined, the other common parameters can be determined graphically by drawing a perpendicular to the **THL** through the **CSXY**. The intersection with the **THL** is the **pp**. The line from the **pp** to the **CSXY** is the **f′**, and by definition the tilt **t** is 90°. The swing **s** is computed according to the alignment of the focal length line (principal line) and the measuring y-axis. The azimuth **a** is computed independently for each crate, and can be determined graphically as shown in Figure 4. The analytical solutions for these Phase-1 parameters are **f′** = R1 sin(Ang3) (check using **f′** = R2 sin(Ang1)), xpp = R1 cos(180 − Ang3) (check using xpp = c2x − R2 cos(Ang1)), and ypp equals the y value of **THL**. The angles (Ang1, Ang3) are determined from the law of cosines. These equations are developed from the geometry shown in Figure 4. The user should be aware that the geometry depends on the position of C1 and C2 relative to the **pp**.

In this example, the solution of the multiple two-point perspective was developed using the fact that the two sets of vanishing points had a common **THL**. A solution is also possible with two objects imaged in two-point perspective, but not having a common **THL**. In this case each set of **VPs** defines a different **THL**, and the intersection of the two **THLs** is the **pp**. (Note: Because the **pp** is always the same, it is impossible to have parallel **THLs** in multiple two-point perspective.) The procedures for determining the Phase-1 parameters are completed using each THL independently. The graphical solution for this example is shown in Figure 5.

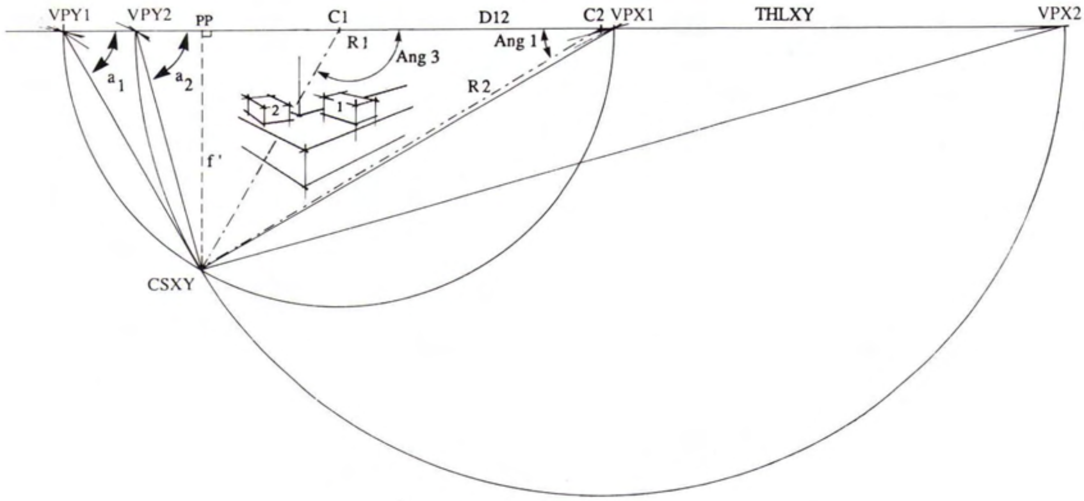

FIG. 4. Multiple two-point perspective solution.

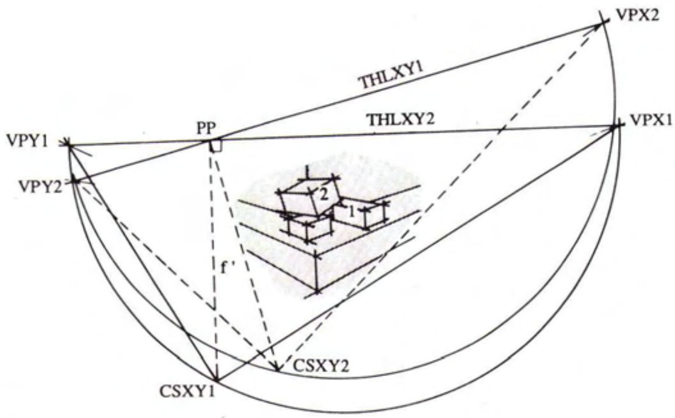

FIG. 5. Multiple two-point perspective — non-parallel **THL**s and a common **pp**.

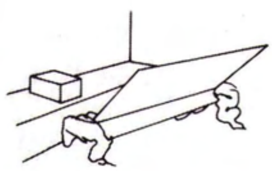

FIG. 6. Combined two- and three-point perspective example.

Once all the Phase-1 parameters are known for the image with the two crates, it is then possible to complete the standard procedures (Phase-2) for determing the object space coordinates and dimensions (Williamson and Brill, 1987).

SINGLE IMAGE WITH COMBINED TWO- AND THREE-POINT PERSPECTIVE

Consider a cropped image containing one crate on the loading dock, and a trailer next to the loading dock (Figure 6). The trailer is neither parallel nor perpendicular to the ground plane of the building or loading dock. On the bed of the trailer is a very large sign, centered on and coincident with the central axis plane of the trailer. The sign has three-point perspective geometry, but — as with the roof of the Cape Cod house — only two of the three vanishing points can be located. A major problem with three-point perspective is to have enough parallel lines to establish each vanishing point. Imagery with combined perspective geometry eliminates that problem, as the two-point solution provides information required by the three-point solution.

This example poses the problem of choosing coordinate systems for both image- and object-space. For simplicity of calculations it is best to transform the image coordinate system, as soon as possible, so the two-point VPY_2 becomes the new origin and the $\mathbf{THL}_2$ becomes the x-axis.

Combined two- and three-point perspective calculations also require common object-space coordinates. Unlike three-point perspective, two-point perspective involves two vertical planes (XZ and YZ). Hence, the object-space coordinate systems imposed by two- and three-point geometries are shifted and rotated relative to one another. Both the two- and three-point coordinate systems are orthogonal right-handed systems. Combining the systems requires using one object-space coordinate system to define point locations in both object-space systems. Ideally, this is developed using the projective equations and a point common to both geometry sets. A form of the projective equations, shown in Equation 7, provides the object-space coordinates of points in a system whose origin is the camera station, whose Z axis is along the principal ray, and whose X and Y axes are parallel to the image coordinate axes x and y.

$$\begin{vmatrix} X \\ Y \\ Z \end{vmatrix}_{\mathrm{IP}} = [\mathbf{R}] \begin{vmatrix} X - Xc \\ Y - Yc \\ Z - Zc \end{vmatrix} \tag{7}$$

The subscript IP stands for image plane. The projective equations for the three-point solution are written as

$$\begin{vmatrix} X_j \\ Y_j \\ Z_j \end{vmatrix}_{IP} = \begin{vmatrix} x_j - xpp \\ y_j - ypp \\ 0 - f' \end{vmatrix} k_j{}^{-1}{}_{3p} = [\mathbf{R}]_{3p} \begin{vmatrix} X_j - Xc \\ Y_j - Yc \\ Z_j - Zc \end{vmatrix}_{3p} \tag{8}$$

The subscript $3p$ defines the variables involved as belonging to the three-point solution. (To define two-point terms, $2p$ may be substituted for $3p$ in Equation 8.) Values on the left side of Equation 8 are the same for the two- and three-point solutions. All the terms on the right side of Equation 8 are Phase-1 and Phase-2 parametric values.

To convert the three-point object-space coordinates to two-point object-space coordinates, combine the two- and three-point versions of Equation 8 to obtain

$$\begin{vmatrix} X_j \\ Y_j \\ Z_j \end{vmatrix}_{2p} = \begin{vmatrix} Xc \\ Yc \\ Zc \end{vmatrix}_{2p} + [\mathbf{R}]^{T}{}_{2p}\,[\mathbf{R}]_{3p} \begin{vmatrix} X_j - Xc \\ Y_j - Yc \\ Z_j - Zc \end{vmatrix}_{3p}. \tag{9}$$

The procedure for using Equation 9 is simple. The Phase-1 and Phase-2 parametric values for both the two- and three-point solutions have already been found, and the object-space coordinates for the point in the three-point solution are determined. This information determines the right-hand side of Equation 9, and hence the XYZ coordinates in the two-point solution on the left-hand side of Equation 9. The important features of this procedure are (1) Phase-1 and -2 parameters are computed only once, (2) the three-point XYZ values are computed for each point, and (3) the three-point XYZ values cannot be computed without knowing one of the XYZ values.

The step-by-step procedures for completing the photogrammetric analysis are discussed in the following paragraphs. Although very tedious if done using a non-programmable desktop calculator, the procedures can easily be programmed for desktop computers, and are streamlined by using a digitizer to collect the image data.

The required Phase-1 parameters for two- and three-point perspective analysis are the principal point coordinates (xpp, ypp); the effective focal length (**f**′); the rotation matrix [**R**] involving **a**, **t**, and **s**; vanishing point VPX(x,y) for the X-axis; vanishing point VPY(x,y) for the Y-axis; and vanishing point VPZ(x,y) for the Z-axis ($3p$ only). These values are presumed to be found by methods previously discussed in this paper.

Now assume that a known vertical distance (**V**) is to be used in both the two- and three-point solutions to determine the object-space camera station **CS** (a Phase-2 computation). Because the distance **V** is known, the coordinates of the endpoints can be assigned, e.g., to ($X1$, $Y1$, $Z1$) and ($X2$, $Y2$, $Z2$), where $X1 = X2$, $Y1 = Y2$, and $Z2 - Z1$ = **V**. Any other known dimension, in the orthogonal planes, may be substituted and the coordinates assigned accordingly. For instance, a horizontal dimension (**H**) in the X or Y direction could be used, and $X2 - X1$ = **H** or $Y2 - Y1$ = **H**. However, when the dominant geometry of the image is two-point, it is best to use a vertical line, which is represented true-view (not foreshortened by perspective).

The equations for the **CS** are defined in three cases – XY, XZ, and YZ – depending on which plane contains the known dimension. In the Appendix (Section V), a procedure is listed to determine the two- and three-point camera station coordinates (Xc_2, Yc_2, Zc_2), and (Xc_3, Yc_3, Zc_3). Because there are two different coordinate systems, the object-space coordinates for the camera stations will have different origins and orientations for the two-point and three-point solutions.

Two-Point Solution in the Combined Image

For the two-point analysis without full format, the origin of the mensuration coordinates is best selected such that mensuration values are positive. The following steps determine the Phase-1 values for the two-point perspective of Figure 6 (using only the right-hand crate), given a known vertical-diagonal angle. The application procedures are elaborated in the Appendix.

Step 1: Locate **VPX**. Using three or more parallel lines, measure two points per line, and calculate the image coordinates of the **VPX** (VPXx, VPXy).
Step 2: Locate **VPY**. Select three or more lines, and using the same method as in Step 1, determine the coordinates of the **VPY** (VPYx, VPYy).
Step 3: Transform image coordinate system so **VPY** becomes the new origin. Use the transformation procedures in the Appendix, Section IV, with the x-axis defined coincident to the **THL**, and $+x$ towards **VPX**. Determine the coordinates of the midpoint C between **VPX** and **VPY** on the **THL**.
Step 4: Locate vertical for VPDVX. Using **VPX-C** as one leg of a 45° right triangle, calculate the endpoint of the end of the leg (vertical side) of the right triangle. The vertical side will be a line through **VPX** perpendicular to **THLXY**. The **VPDVX** will be located on this line.
Step 5: Locate **VPDVX**. Extend the known image diagonal of the crate until it intersects with the vertical line of Step 4. Use the two-line intersection equations to determine the intersection, which is the vanishing point **VPDVX**, with image coordinates VPDVXx, VPDVXy. (If several known diagonals are available, then as many **VPDVX** positions should be determined, which in turn will help determine the location of **MPX**.)
Step 6: Locate **MPX** (**CSXY** rotated). With the position of **VPDVX** and the true angle of the diagonal known, calculate **MPX** using MPXx = VPXx – (VPDVXy – VPXy) tan (angle), and MPXy = 0.
Step 7: Locate **CSXY**. The graphical location of **CSXY** involves rotating the point MPX, with **VPX** as the center of rotation, to intersect with the semicircle for **VPX-VPY**. Analytically, the solution involves using the isosceles triangle **CSXY** – C1 – **VPX** with the radius of the semicircle as the two equal sides and the midpoint (C1) of **VPX-VPY** as the apex. The side opposite C1 is **VPX-CSXY**, which is equal to the line **VPX-MPX**. The angle (Angl) at **VPX** is determined using the law of cosines, and the sides of the isosceles triangle. The **CSXY** coordinates are computed using CSXYx = VPXx – (**VPXx** – **MPXx**)cos(Angl), and CSXYy = – (**VPXx** – **MPXx**)sin(Angl).
Step 8: Determine remaining Phase-1 parameters. The **pp**, **f**′, **a**, and **s** can now be determined from the coordinates of **VPX**, **VPY**, and **CSXY**. The tilt angle, by definition, is 90°. The swing angle will be 180°, if the image coordinate axis for x is parallel to both the **THL** and the bottom edge of the frame. Equations to determine the rotation matrix are listed in Section VI of the Appendix, and equations for the angles are in Section VII.
Step 9: Locate **XC**, **YC**, **ZC**. Given the two-point perspective Phase-1 parameters, determine the Phase-2 camera station coordinates. First compute one **CS** coordinate (using Section V of Appendix). Next compute the remaining two coordinates (using Section VIII of Appendix). (For two-point perspective, the value of | VPZy | should be an extremely large number, e.g., $1 \times 10^{+7}$, instead of zero, and the **t** will be 90°.) Of course, there are also other ways to determine the object-space camera station coordinates.

Three-Point Solution in the Combined Image

The procedures for the three-point solution, of Figure 6, are a continuation of the two-point steps.

Step 10: Locate **VPX$_3$**. Compute the image coordinates of **VPX$_3$** using the same procedures described in Step 1.
Step 11: Locate **VPZ$_3$**. Using the same procedures of Step 1, to determine the image coordinates of **VPZ$_3$**.
Step 12: Locate **VPY$_3$**. With the known image coordinates of **pp**, **VPX$_3$**, and **VPZ$_3$**, compute **VPY$_3$** using the equations in Section XI of the Appendix.
Step 13: Locate the **f′$_3$**. Determine the effective focal length (**f**′) for the three-point solution by using one of more of the focal-length equations listed in Section IX of the Appendix.
Step 14: Compute [**R**]. Calculate the rotation matrix elements using the same procedures described in Step 8.
Step 15: Locate **XC**, **YC**, **ZC** for three-point geometry, and find the camera station coordinates for the three-point solution as in Step 9.
Step 16: Locate an object-space point. Given a known object-space coordinate of a point in the three-point solution, compute the other two coordinates of this point, using equations in Section X of the Appendix.
Step 17: Convert the object-space coordinates of Step 16 to two-point object-space using Equation 11.
Step 18: Repeat steps 16 and 17 to convert other three-point object-space coordinates.

This completes the analytical procedures for the example of combined two- and three-point perspective. The result of the graphical procedure is shown in Figure 7.

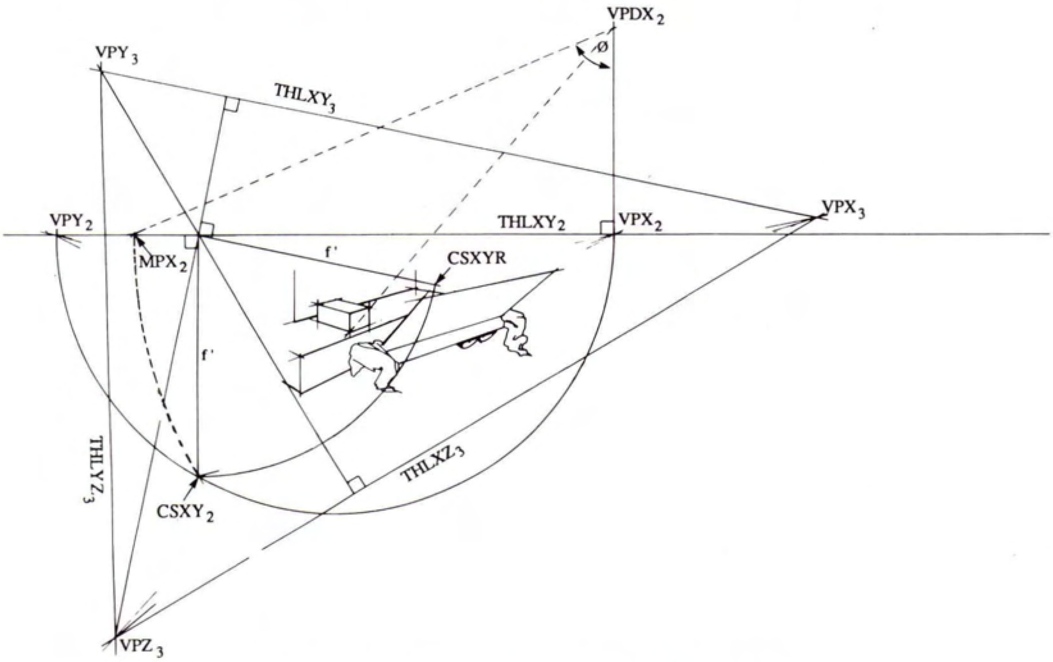

FIG. 7. Combined two- and three-point perspective solution.

SUMMARY

Although three-dimensional reconstruction from a single image always requires some prior information about object space, less information is needed when there is multiple or combined dominant geometry than when the geometry is simple. To illustrate this fact, we chose a multiple two-point perspective in which the imaged objects rest on a common horizontal plane. In cropped imagery with simple two-point perspective, a horizontal or vertical diagonal angle had to be known to complete the reconstruction (Williamson and Brill, 1987). However, such an angle need not be known for cropped imagery with multiple two-point perspective in which nonparallel objects share the same ground plane or share parallel ground planes. Under these conditions the **CSXY** is found by locating the vanishing points for each nonparallel object, constructing the circles defined by the respective sets of vanishing points, and identifying an intersection point of the circles as the **CSXY**. With three or more sets of vanishing points, the precision of the method is indicated by how close the circles come to intersecting at a point. If the respective object horizontal planes were not parallel, but each offered two-point perspective, the camera station could emerge with equal ease by noting that the unique principal point is the intersection between the **THL**s.

Nonparallel ground planes that are perpendicular to the image plane are rare. More frequent are planes that are skewed both to the two-point horizontal plane and to the image plane (as in three-point perspective). However, as was the case with the roof of the Cape Cod house and the sign on the trailer, such planes are often not attached to deep enough rectangular structures to enable determination of more than two vanishing points. It has been shown here that available two-point geometry can be used to augment the three-point problem to retrieve the missing three-point vanishing point. As in most combined-perspective geometry problems, there is a need to embed the solutions in a single object-space coordinate system. This can be done analytically using a dimension of known length parallel to one of the major orthogonal coordinate planes.

From the examples discussed, it should be clear that combined perspective geometries in a single image can serve not only to fill in missing information, but also to strengthen geometry by using redundant information to enhance the precision of the solution. In this paper, emphasis was given to some analytical representations, with recommendations that the procedures be placed on a programmable calculator or desktop computer system for use in a close-range photogrammetric workstation. Such a workstation, developed for multiple close-range digital imagery (Williamson, et al, 1988), incorporates these single image-perspective procedures as a secondary system.

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APPENDIX

The following eleven application procedures were taken from a book (in preparation by the authors) on analytical and graphical procedures for the photogrammetrist.

I. Intersection of Two Lines

$W_1 = (y_{A1}-y_{A2})(x_{B2}-x_{B1}) - (y_{B1}-y_{B2})(x_{A2}-x_{A1})$
$W_2 = (y_{A1}x_{A2}-y_{A2}x_{A1})$, $W_3 = (y_{B1}x_{B2}-y_{B2}x_{B1})$
$VPx = (W_2(x_{B2}-x_{B1})-W_3(x_{A2}-x_{A1}))/W_1$
$VPy = (W_3(y_{A1}-y_{A2})-W_2(y_{B1}-y_{B2}))/W_1$

II. Intersection of Multiple Lines, with Minimum Distance

The following equations (program code) are used to establish the matrix of coefficients to be used in a Gaussian simultaneous adjustment routine. In this code the terms X(N), Y(N), X(NP1), and Y(NP1) are the coordinates of two points on a line. The number of lines read is N/2. D(K) is the minimum distance of each line to the intersection point XI, YI, and the number of lines is NMAX.

```
   DO 10 N = 1, NMAX, 2
     NP1 = N + 1
     READ * X(N),Y(N),X(NP1),Y(NP1)
     Q(1) = Y(NP1) - Y(N)
     Q(2) = X(N) - X(NP1)
     Q(3) = Y(N)*X(NP1) - X(N)*Y(NP1)
     DO 10 I = 1, 2
       DO 10 J = 1, 3
       C(I,J) = C(I,J) + Q(I)*Q(J)
10 CONTINUE
   CALL [GAUSSIAN ROUTINE](C,2)
   XI = C(1,3)
   YI = C(2,3)
   WRITE * XI,YI
   K = 0
   DO 20 M = 1, NMAX, 2
     K = K + 1
     MP1 = M + 1
     Q(1) = Y(MP1) - Y(M)
     Q(2) = X(M) - X(MP1)
     Q(3) = Y(M)*X(MP1) - X(M)*Y(MP1)
     S = SQRT(Q(1)**2 + Q(2)**2)
     D(K) = (Q(1)*XI + Q(2)*YI + Q(3))/S
     WRITE * D(K)
20 CONTINUE
```

III. Multiple Two-Point Perspective Solution for CS

Locate C1 and C2:
$C1x = (VPX1x + VPY1x)/2$, $C1y = (VPX1y + VPY1y)/2$
$C2x = (VPX2x + VPY2x)/2$, $C2y = (VPX2y + VPY2y)/2$
Determine the radii and distance between centers
$R1 = [(VPX1x-VPY1x)^2+(VPX1y-VPY1y)^2]^{1/2}/2$
$R2 = [(VPX2x-VPY2x)^2+(VPX2y-VPY2y)^2]^{1/2}/2$
$D12 = [(C1x-C2x)^2+(C1y-C2y)^2]^{1/2}$
Solve for **CSXY** using Law of Cosines
$Ang1 = \mathrm{atan}[(C2y-C1y)/(C1x-C2x)]$
$Ang2 = \mathrm{acos}[(D12^2+R2^2-R1^2)/2(R2\ D12)]$
$Ang3 = Ang1+Ang2$
$CSXYx = C1x + R1\cos(Ang3)$
$CSXYy = C1y - R1\sin(Ang3)$

IV. Transformation of Image Origin to VPY, x-Axis on THL

$w1 = VPYy - VPXy$, $w2 = VPXx - VPYx$
$xt = x_i - VPYx$, $yt = y_i - VPYy$
$x_i = (xt\ w1 - yt\ w2)/(w1^2 + w2^2)^{1/2}$
$y_i = (xt\ w2 + yt\ w1)/(w1^2 + w2^2)^{1/2}$

V. Compute a CS Coordinate from a Distance in Object-Space

One object-space camera coordinate can be computed using a known dimension parallel to a coordinate plane (*XY*, *XZ*, or *YZ*). The known dimension is defined as **DXY, DXZ,** or **DYZ,** depending on the coordinate plane to which it is parallel. All Phase-1 parametric values for the image must be known.

$wx_i = x_i - xpp$, $wy_i = y_i - ypp$
where $i = 1,2$ are the end points to known line;
use $i = 1$ values
$w1 = r_{11}\,wx_i + r_{21}\,wy_i - r_{31}f'$
$w2 = r_{12}\,wx_i + r_{22}\,wy_i - r_{32}f'$
$w3 = r_{13}\,wx_i + r_{23}\,wy_i - r_{33}f'$
Repeat eqs. $w1$ to $w3$ as $w4$ to $w6$, using $i = 2$.

Case One: Given distance (DXY) is parallel to *XY* plane:
$w7 = (w4/w6) - (w1/w3)$, $w8 = (w5/w6) - (w2/w3)$
$w9 = [w7^2 + w8^2]^{1/2}$. If $Z1<ZC$, then $w9 = -w9$.
(Where end point 1 is below the horizon line)
Given X1: $XC = X1-(w1/w3)\ DXY/w9$
Given Y1: $YC = Y1-(w2/w3)\ DXY/w9$
Given Z1: $ZC = Z1-(DXY/w9)$

Case Two: Given distance (DXZ) is parallel to *XZ* plane:
$w7 = (w4/w5) - (w1/w6)$, $w8 = (w6/w5) - (w3/w2)$
$w9 = [w7^2 + w8^2]^{1/2}$. If $Y1<YC$, then $w9 = -w9$.
Given X1: $XC = X1-(w1/w2)\ DXZ/w9$
Given Y1: $YC = Y1-(DXZ/w9)$
Given Z1: $ZC = Z1-(w3/w2)\ DXZ/w9$

Case Three: Given distance (DYZ) is parallel to *YZ* plane:
$w7 = (w5/w4) - (w2/w1)$, $w8 = (w6/w4) - (w3/w1)$
$w9 = [w7^2 + w8^2]^{1/2}$ If $X1<XC$, then $w9 = -w9$
Given X1: $XC = X1-(DYZ/w9)$
Given Y1: $YC = Y1-(w2/w1)\ DYZ/w9$
Given Z1: $ZC = Z1-(w3/w1)\ DYZ/w9$

VI. Compute [R]

Let $i = 1$ for VPX, $i = 2$ for VPY, and $i = 3$ for VPZ.
$wX_i = VPx_i - xpp$, $wY_i = VPy_i - ypp$
$VR_i = [wX_i^2 + wY_i^2 + f'^2]^{1/2}$
$r_{1i} = wX_i/VR_i$, $r_{2i} = wY_i/VR_i$, $r_{3i} = -f'/VR_i$
Note: If $w6$ is negative, then $r_{13} = -r_{13}$, $r_{23} = -r_{23}$, $r_{33} = -r_{33}$.

VII. Compute the Rotation Angles Using [R]

Angles are computed in decimal degrees.
$a = \tan^{-1}(\mathrm{abs}(r_{31}/r_{32}))$ [0° to 90°]
$t = \cos^{-1}(r_{33})$ [0° to 180°]
$s = \tan^{-1}(r_{13}/r_{23}) + 180.0$ [90° to 270°]

VIII. Compute the Remaining Two CS Coordinates

Start by computing $w1$, $w2$, and $w3$ as in Section V.

Case One: Known Camera Station Coordinate is XC.
$YC = Y1 + (w2/w1)\ (XC-X1)$
$ZC = Z1 + (w3/w1)\ (XC-X1)$

Case Two: Known Camera Station Corrdinate is YC.
$XC = X1 + (w1/w2)\ (YC-Y1)$
$ZC = Z1 + (w3/w2)\ (YC-Y1)$

Case Three: Known Camera Station Coordinate is ZC.

$XC = X1 + (w1/w3)\ (ZC - Z1)$
$YC = Y1 + (w2/w3)\ (ZC - Z1)$

IX. Compute the Effective Focal Length (f')

The value of f' is determined from one or more of these vector dot-product equations:

$$f'^2 = |(VPX - PP)\cdot(VPY - PP)|$$
$$= |(VPY - PP)\cdot(VPZ - PP)|$$
$$= |(VPZ - PP)\cdot(VPX - PP)|$$

The application equations are:

$f'xy = [(VPXx - xpp)(VPYx - xpp) + (VPXy - ypp)(VPYy - ypp)]^{1/2}$
$f'xz = [(VPXx - xpp)(VPZx - xpp) + (VPXy - ypp)(VPZy - ypp)]^{1/2}$
$f'yz = [(VPYx - xpp)(VPZx - xpp) + (VPYy - ypp)(VPZy - ypp)]^{1/2}$

where f' may be one or the average of these equations.

X. Compute the Remaining Two Object-Space Coordinates

Start by computing $w1$, $w2$, and $w3$ as in Section V.

Case One: Known Object-Space Corrdinate is Z1.

$X1 = Xc + (w1/w3)\ (Z1 - Zc)$
$Y1 = Yc + (w2/w3)\ (Z1 - Zc)$

Case Two: Known Object-Space Coordinate is Y1.

$X1 = Xc + (w1/w2)\ (Y1 - Yc)$
$Z1 = Zc + (w3/w2)\ (Y1 - Yc)$

Case Three: Known Object-Space Coordinate is X1.

$Y1 = Yc + (w2/w1)\ (X1 - Xc)$
$Z1 = Zc + (w3/w1)\ (X1 - Xc)$

XI. Determine VPY Given pp and other Two VPs.

$w1 = VPZy\ (ypp - VPXy) + VPZx\ (xpp - VPXx)$
$w2 = ypp\ (VPZy - VPXy) + xpp\ (VPZx - VPXx)$
$w3 = w1\ (VPZy - VPXy) - w2\ (ypp - VPXy)$
$w4 = w1\ (VPZx - VPXx) - w2\ (xpp - VPXx)$
$w5 = (ypp - VPXy)(VPZx - VPXx) - (VPZy - VPXy)(xpp - VPXx)$
$VPYx = w3 / w5$
$VPYy = w4 / w5$