# The Duality and Critical Condition in the Formulation and Decomposition of a Rotation Matrix 

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#### Abstract

The formulation and decomposition of a rotation matrix with physical elements, i.e., the rotation angles and algebraic elements, are described. The dual set of rotation elements and critical conditions for the decomposition of a rotation matrix are studied.


## INTRODUCTION

BOTH IN PHOTOGRAMMETRY AND GEODESY, as well as in other fields related to geometry, the rotation matrix in a threedimensional (3-D) Cartesian coordinate system has an important role. A rotation matrix pertains orthogonal properties. This means that

$$
\begin{equation*}
\mathrm{A}^{\top} \mathrm{A}=\mathbf{I} \tag{1}
\end{equation*}
$$

or, in other words,

$$
\begin{equation*}
\mathbf{A}^{\mathrm{T}}=\mathrm{A}^{-1} \tag{2}
\end{equation*}
$$

Based on the rotation angles defined by the axes of the coordinate system, the rotation matrix between two 3-D systems can be formulated by a series of matrix multiplications of the rotation matrices of each individual element.

There are a number of ways to define the rotation angles in a 3-D Cartesian coordinate system, and there are also several different sequences for the matrix multiplication. The first class includes the ( $\omega, \phi, \kappa$ ) system, which is widely applied in aerial photogrammetry, and the ( $\alpha, t, s$; azimuth, tilt, swing) system, which has its major application in oblique photography. The second class includes, e.g., $(\omega-\phi-\kappa)$ and $(\phi-\omega-\kappa)$ sequences, which is usually referred to as the order of the rotation matrix. The different orders of the rotation matrix results from the well-known fact of non-commutativity of matrix multiplication, i.e., for two multiplicable matrices,

$$
A B \neq B A
$$

unless A or B is an identity or null matrix.
Apart from these methods of forming the rotation matrix by physically defined elements, other mathematical formulations of orthogonal matrices can also be applied for computational operations. Schut (1959) is referred to for an extensive review of the algebraic formulation of orthogonal matrices. For applications in photogrammetry, two algebraic formulations have received much more attention than others, namely, those formulated from the elements of a skew-symmetric matrix and those formulated from the components of a unit quaternion. The first scheme has been utilized by Ackermann in the program PAT-M (Ackermann et al., 1973) while the second one has been applied by Pope for the American NGS programs (Pope, 1970) and by Schut for Canadian NRC programs (Schut, 1978).

## THE COORDINATE SYSTEM

We define the coordinate system as a three-dimensional rectangular right-handed coordinate system. The positive directions of the angles are defined as the clockwise direction viewed from the origin of the coordinate system to the positive direction of each axis (Figure 1) (Thompson, 1969). Therefore, when referred to the $(X-Y),(Y-Z),(X-Z)$ planes, the positive directions of rotating a point in a fixed coordinate system are as shown in Figure 2.


Fig. 1. The coordinate system.


Fig. 2. The positive angles.

## THE ROTATION MATRIX WITH PHYSICAL ELEMENTS

## The Formulation

Based on the above definition, the rotation matrix of a single element can be formulated as $\mathbf{R}_{x}, \mathbf{R}_{y}, \mathbf{R}_{z}$, with respect to the rotation around the $X, Y, Z$ coordinate axis, respectively.

Let

$$
\begin{aligned}
p & =(x, y, z) \text { before the rotation, and } \\
p^{\prime} & =\left(x^{\prime}, y^{\prime}, z^{\prime}\right) \text { after the rotation }
\end{aligned}
$$

for the rotation around the $X$-axis (see Figure 3). Then

$$
\begin{aligned}
& x^{\prime}=x \text { and } \\
& y^{\prime}=d[\cos (\omega+\theta)]=d[\cos \omega \cos \theta-\sin \omega \sin \theta]=y \cos \omega-z \sin \omega .
\end{aligned}
$$

Representing this in matrix form, we obtain

$$
\begin{align*}
& \mathbf{R}_{X}=\mathbf{R}_{\omega}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \omega- & \sin \omega \\
0 & \sin \omega & \cos \omega
\end{array}\right), \\
& \mathbf{R}_{\gamma}=\mathbf{R}_{\phi}=\left(\begin{array}{ccc}
\cos \phi & 0 & \sin \phi \\
0 & 1 & 0 \\
-\sin \phi & 0 & \cos \phi
\end{array}\right),  \tag{3}\\
& \mathbf{R}_{Z}=\mathbf{R}_{\kappa}=\left(\begin{array}{ccc}
\cos \kappa & -\sin \kappa & 0 \\
\sin \kappa & \cos \kappa & 0 \\
0 & 0 & 1
\end{array}\right) .
\end{align*}
$$

Applying matrix multiplication by the order ( $\omega-\phi-\kappa$ ) and ( $\phi-\omega-\kappa$ ), we obtain

$$
\begin{align*}
& \mathbf{R}_{\omega \phi \kappa}=\left(\begin{array}{ccc}
\cos \phi \cos \kappa & -\cos \phi \sin \kappa & \sin \phi \\
\cos \omega \sin \kappa+\sin \omega \sin \phi \cos \kappa & \cos \omega \cos \kappa-\sin \omega \sin \phi \sin \kappa & -\sin \omega \cos \phi \\
\sin \omega \sin \kappa-\cos \omega \sin \phi \cos \kappa & \sin \omega \cos \kappa+\cos \omega \sin \phi \sin \kappa & \cos \omega \cos \phi
\end{array}\right),  \tag{4}\\
& \mathbf{R}_{\phi \text { ток }}=\left(\begin{array}{ccc}
\cos \phi \cos \kappa+\sin \phi \sin \omega \sin \kappa & -\cos \phi \sin \kappa+\sin \phi \sin \omega \cos \kappa & \sin \phi \cos \omega \\
\cos \omega \sin \kappa & \cos \omega \cos \kappa & -\sin \omega \\
-\sin \phi \cos \kappa+\cos \phi \sin \omega \sin \kappa & \sin \phi \sin \kappa+\cos \phi \sin \omega \cos \kappa & \cos \omega \cos \phi
\end{array}\right) \tag{5}
\end{align*}
$$

The above two rotation matrices are those for the case of rotating a point with respect to a fixed coordinate system. If the point in fixed and the coordinate system is rotating, then the rotation matrix is $\mathbf{M}=\mathbf{R}^{\mathbf{T}}$. It is also worth noting that

$$
\begin{align*}
\mathbf{M}_{\omega \prime} & =\mathbf{R}_{\omega,}^{\top}, \\
\mathbf{M}_{\phi} & =\mathbf{R}_{\phi,}^{\top} \\
\mathbf{M}_{\kappa} & =\mathbf{R}_{\kappa}^{\top} .  \tag{6}\\
\mathbf{M}_{\omega \omega \kappa \kappa} & =\mathbf{M}_{\kappa} \mathbf{M}_{\phi} \mathbf{M}_{\omega}=\mathbf{R}_{\kappa}^{\top} \mathbf{R}_{\phi}^{\top} \mathbf{R}_{\omega}^{\top}=\left(\mathbf{R}_{\omega} \mathbf{R}_{\phi} \mathbf{R}_{\kappa}\right)^{\top}=\mathbf{R}_{\omega \phi \kappa}^{\top} \\
\mathbf{M}_{\phi \omega \kappa} & =\mathbf{M}_{\kappa} \mathbf{M}_{\omega} \mathbf{M}_{\phi}=\mathbf{R}_{\kappa}^{\top} \mathbf{R}_{\omega}^{\top} \mathbf{R}_{\phi}^{\top}=\left(\mathbf{R}_{\omega} \mathbf{R}_{\omega} \mathbf{R}_{\kappa}\right)^{\top}=\mathbf{R}_{\phi \omega \kappa}^{\top}
\end{align*}
$$

For the ( $\alpha, t, s$ ) system (Figure 4), the resulting rotation matrix is

$$
\mathbf{R}_{\alpha t s}=\left(\begin{array}{ccc}
-\cos s \cos \alpha-\sin s \cos t \sin \alpha & \sin s \cos \alpha-\cos s \cos t \sin \alpha & -\sin t \sin \alpha  \tag{7}\\
\cos s \sin \alpha-\sin s \cos t \cos \alpha & \sin s \sin \alpha-\cos s \cos t \cos \alpha & -\sin t \cos \alpha \\
-\sin s \sin t & -\cos s \sin t & \cos t
\end{array}\right) .
$$

## The Decomposition

Decomposition of a rotation matrix into the rotation elements can be performed by investigating correspondent trigonometric function values of that element. For $\mathbf{R}_{\omega \phi \kappa \kappa}$, the $\kappa$ can be obtained from $R_{11}$ and $R_{12}$. If we divide $R_{12}$ by $R_{11}$, and then take the arc-
tangent of the result, the $\kappa$ value can be recovered. However, there is the sign problem, because both $\theta$ and $(\theta \pm \pi)$ can satisfy the arc-tangent function (Figure 5). This ambiguity can be resolved by studying the algebraic $\operatorname{signs}$ of $\cos \theta$ and $\sin \theta$. In the programming language FORTRAN, this utility is well-known as ATAN $2(y, x)$, which takes $(y, x)$ as input arguments. In our case, the sign of $\cos \phi$ complicates the problem. If $\cos \phi$ is positive, then $\kappa$ should be ATAN $2\left(-R_{12}, R_{11}\right)$. If $\cos \phi$ is negative, $\kappa$ should be $\operatorname{ATAN} 2\left(R_{12},-R_{11}\right)$. Thus, when $\cos \phi$ is positive, then

$$
\begin{align*}
& \kappa=\operatorname{ATAN} 2\left(-R_{12}, R_{11}\right) ; \\
& \phi=\operatorname{ATAN} 2\left(R_{13}, R_{11} / \cos \kappa\right) ;  \tag{8}\\
& \omega=\operatorname{ATAN} 2\left(-R_{23}, R_{33}\right)
\end{align*}
$$

If, in fact, $\cos \phi$ is negative, then the resulting angles will be $(\pi+\omega, \pi-\theta, \pi+\kappa)$ respectively. This set of angles in fact formulates the same rotation matrix as ( $\omega, \phi, \kappa$ ). In other words, there are two sets of rotation elements which will form the same rotation matrix by the same order of rotating sequence.

For $\mathbf{R}_{\phi \text { ouk }}$ the second set of rotation elements is $(\pi-\omega, \pi+\phi, \pi+\kappa)$. The solution scheme is
$\cos \omega$ is positive

$$
\begin{align*}
& \kappa=\operatorname{ATAN} 2\left(R_{22}, R_{12}\right) ; \\
& \phi=\operatorname{ATAN} 2\left(R_{13}, R_{33}\right) ;  \tag{9}\\
& \omega=\operatorname{ATAN} 2\left(-R_{23}, R_{21} / \sin \kappa\right) .
\end{align*}
$$

In the practical computation, several schemes may be applied. One of these utilizes the first-order approximation matrix. For either $\mathbf{R}_{\omega \phi k}$ or $\mathbf{R}_{\phi \omega \kappa}$, if the angles are fairly small, then $\sin \theta$ will approach $\theta$ and $\cos \theta$ will approach 1 . Substituting these into either $\mathbf{R}_{\omega \phi \kappa}$ or $\mathbf{R}_{\text {фок }}$ and neglecting the second order terms of the sine functions, the rotation matrix will be reduced to

$$
\mathbf{R}_{\omega \phi \kappa}=\mathbf{R}_{\phi \omega \omega}=\left(\begin{array}{crr}
1 & -\kappa & \phi  \tag{10}\\
\kappa & 1 & -\omega \\
-\phi & \omega & 1
\end{array}\right)
$$

This indicates that, when the rotation elements are of small magnitude, the value and sign of ( $\omega, \phi, \kappa$ ) will be close to $R_{32}, R_{13}$, $R_{21}$ ). This can yield a unique solution. This actually results from the fact that $\cos \theta \in R^{+}$, if $|\theta|<90^{\circ}$. That is, if we specify $\phi$ to be less than $90^{\circ}$, then there is a unique solution. However, for the special case, i.e., $\cos \phi=0$ and $\sin \phi=1$, the rotation matrix $\mathbf{R}_{\omega \phi \kappa}$ will be reduced to

$$
\mathbf{R}_{\omega \phi \kappa}=\left(\begin{array}{ccc}
0 & 0 & 1  \tag{11}\\
\cos \omega \sin \kappa+\sin \omega \cos \kappa & \cos \omega \cos \kappa-\sin \omega \sin \kappa & 0 \\
\sin \omega \sin \kappa-\cos \omega \cos \kappa & \sin \omega \cos \kappa+\cos \omega \sin \kappa & 0
\end{array}\right)
$$



FIG. 3. Illustration for the rotations and coordinates.


FIG. 4. Definition of the $(\alpha, s, t)$ system.


FIG. 5. The sign of sine and cosine functions.

This can be further represented as

$$
\mathbf{R}_{\omega \phi \kappa}=\left(\begin{array}{ccc}
0 & 0 & 1  \tag{12}\\
\sin (w+\kappa) & \cos (\omega+\kappa) & 0 \\
-\cos (\omega+\kappa) & \sin (\omega+\kappa) & 0
\end{array}\right) .
$$

Apparently, the proposed decomposition procedures do not work. In fact, there are an infinite number of solutions. It is interesting to note that, for $\mathbf{R}_{\omega \phi \kappa}$, the angles $\omega$ and $\kappa$ do not have the same characteristics on rotation matrix decomposition. For $\mathbf{R}_{\text {dow }}$ the key element is $\omega$, if $\cos \omega=0$ and $\sin \omega=1$, then

$$
\mathbf{R}_{\text {фowк }}=\left(\begin{array}{ccr}
\cos (\phi-\kappa) & \sin (\phi-\kappa) & 0  \tag{13}\\
0 & 0 & -1 \\
-\sin (\phi-\kappa) & \cos (\phi-\kappa) & 0
\end{array}\right) .
$$

For the ( $\alpha, s, t$ ) system, the decomposition scheme would be:

$$
\begin{align*}
& \text { assume } \sin t \in R^{+} \\
& \alpha=\operatorname{ATAN} 2\left(-R_{13},-R_{23}\right), \\
& t=\operatorname{ATAN} 2\left(-R_{22} / \cos \alpha, R_{33}\right),  \tag{14}\\
& s=\operatorname{ATAN} 2\left(-R_{31},-R_{32}\right) .
\end{align*}
$$

If $\sin t$ is negative, then the resulting elements will be $(\pi+\alpha, \pi+s,-t)$. The key element is the tilt angle $t$. If $\cos t=1$, then $\sin t=0$, and the rotation matrix cannot be decomposed with a finite solution. Physically, it is understood that, when the tilt angle is zero, i.e., the photograph is truly vertical, the two defining planes are either parallel or coincident; thus, the angles of swing and azimuth are undefined (ASP, 1980).

## ORTHOGONAL MATRIX FROM A SKEW-SYMMETRIC MATRIX

## The Formulation

Based on Cayley's formula (also known as Cayley-Klein Formula) if $\mathbf{S}$ is a real skew-symmetric matrix, then

$$
(\mathbf{I}-\mathbf{S})(\mathbf{I}+\mathbf{S})^{-1}
$$

is orthogonal (Thompson, 1969). The resulting orthogonal matrix is sometimes called the Rodrigues matrix.
In photogrammetric applications, there are two formulations resulting from the skew-symmetric matrices:

$$
\mathbf{S}=\left(\begin{array}{rrr}
0 & -c & b \\
c & 0 & -a \\
-b & a & 0
\end{array}\right)
$$

and

$$
\mathbf{S}=\left(\begin{array}{ccc}
0 & -c / 2 & b / 2 \\
c / 2 & 0 & -a / 2 \\
-b / 2 & a / 2 & 0
\end{array}\right)
$$

The first formulation gives

$$
\mathrm{A}=\frac{1}{1+a^{2}+b^{2}+c^{2}}\left(\begin{array}{ccc}
1+a^{2}-b^{2}-c^{2} & 2 a b-2 c & 2 a c+2 b  \tag{15}\\
2 a b+2 c & 1-a^{2}+b^{2}-c^{2} & 2 b c-2 a \\
2 a c-2 b & 2 b c+2 a & 1-a^{2}-b^{2}+c^{2}
\end{array}\right)
$$

while the second one gives

$$
\mathbf{B}=\frac{1}{1+\left(a^{2}+b^{2}+c^{2}\right) / 4}\left(\begin{array}{ccc}
1+\left(a^{2}-b^{2}-c^{2}\right) / 4 & a b / 2-c & a c / 2+b  \tag{16}\\
a b / 2+c & 1+\left(b^{2}-a^{2}-c^{2}\right) / 4 & b c / 2-a \\
a c / 2-b & b c / 2+a & 1+\left(c^{2}-a^{2}-b^{2}\right) / 4
\end{array}\right)
$$

However, there is practically no difference between these two matrices.

## The Decomposition

For matrix A, let

$$
\begin{equation*}
1+a^{2}+b^{2}+c^{2}=d \tag{17}
\end{equation*}
$$

Then the matrix elements can be expressed as

$$
\begin{align*}
a & =\left(A_{32}-A_{23}\right) d / 4 \\
b & =\left(A_{13}-A_{31}\right) d / 4  \tag{18}\\
c & =\left(A_{21}-A_{12}\right) d / 4 .
\end{align*}
$$

From the orthogonal properties, the following relation exists:

$$
\begin{equation*}
\left[\left(1+a^{2}-b^{2}-c^{2}\right)^{2}+(2 a b-2 c)^{2}+(2 a c+2 b)^{2}\right] / d^{2}=1 \tag{19}
\end{equation*}
$$

Developing the above equation, we obtain

$$
\begin{equation*}
1+2 a^{2}+2 b^{2}+2 c^{2}+2 a^{2} c^{2}+2 a^{2} b^{2}+2 b^{2} c^{2}+a^{4}+b^{4}+c^{4}=d^{2} \tag{20}
\end{equation*}
$$

Let

$$
\begin{align*}
a & =k_{a} d \\
b & =k_{b} d  \tag{21}\\
c & =k_{c} d \text { and }
\end{align*}
$$

Substituting Equation 21 into Equation 20, we obtain

$$
\begin{equation*}
1+\beta d^{2}+\alpha d^{4}=0 \tag{22}
\end{equation*}
$$

where $\alpha=2\left(k_{a}{ }^{2} k_{c}{ }^{2}+k_{a}{ }^{2} k_{b}{ }^{2}-k_{b}{ }^{2} k_{c}{ }^{2}\right)+k_{a}{ }^{4}+k_{b}{ }^{4}+k_{c}{ }^{4}$ and $\beta=2\left(k_{a}{ }^{2}+k_{b}{ }^{2}+k_{c}{ }^{2}-1 / 2\right)$.
There are potentially four roots for $d$; however, recalling $1+a^{2}+b^{2}+c^{2}=d$, therefore, $d$ is positive. That is, $d=+\sqrt{d^{2}}$.
There are two potential roots, resulting in two sets of orthogonal rotation elements. In fact, Equations 19 and 20 can be substituted by Equation 17. Directly substituting Equation 21 into Equation 17, we obtain $1+\beta d+\alpha d^{2}=0$ where $\alpha=k_{a}^{2}+k_{b}{ }^{2}+k_{c}{ }^{2}$ and $\beta=-1$.
This gives exactly the same two roots as obtained from Equation 22. However, both of these two roots cannot be satisfactory at the same time. One of the two is the nuisance root caused by the equation of higher order. The correct root can be identified from the test with an off-diagonal element of the rotation matrix. The duality does not exist in this parameterization.

## ORTHOGONAL MATRIX FROM COMPONENTS OF A UNIT QUATERNION

## The Formulation

According to Pope (1970), the formulation is

$$
\mathbf{Q}=\left(\begin{array}{ccc}
s^{2}+p^{2}-q^{2}-r^{2} & 2(p q+r s) & 2(p r-q s)  \tag{23}\\
2(p q-r s) & s^{2}-p^{2}+q^{2}-r^{2} & 2(q r+p s) \\
2(p r+q s) & 2(q r-p s) & s^{2}-p^{2}-q^{2}+r^{2}
\end{array}\right)
$$

where $p^{2}+q^{2}+r^{2}+s^{2}=1$.

## The Decomposition

From Equation 23, we obtain

$$
\begin{align*}
& Q_{13}+Q_{31}=4 p r \\
& Q_{23}+Q_{32}=4 q r \\
& Q_{12}+Q_{21}=4 p q  \tag{24}\\
& Q_{13}-Q_{31}=4 q s \\
& Q_{23}-Q_{32}=4 p s \\
& Q_{12}-Q_{21}=4 r s
\end{align*}
$$

Using Equation 24, we can formulate the following scheme:

$$
\begin{align*}
& p=\frac{Q_{13}+Q_{31}}{4 r} \\
& q=\frac{Q_{23}+Q_{32}}{4 r}  \tag{25}\\
& r^{2}=\frac{\left(Q_{13}+Q_{31}\right)\left(Q_{23}+Q_{32}\right)}{4\left(Q_{12}+Q_{21}\right)}
\end{align*}
$$

for the case $r=0$, we get

$$
\begin{align*}
p^{2} & =\frac{\left(Q_{11}-Q_{22}\right)}{2}  \tag{26}\\
q & =\frac{Q_{13}}{Q_{33}} p
\end{align*}
$$

If $\left(Q_{12}+Q_{21}\right)$ is approaching zero, another scheme can be derived from Equation 24: i.e.,

$$
\begin{align*}
r & =\frac{Q_{13}+Q_{31}}{4 p} \\
q & =\frac{Q_{12}+Q_{21}}{4 p}  \tag{27}\\
p^{2} & =\frac{\left(Q_{13}+Q_{31}\right)\left(Q_{12}+Q_{21}\right)}{4\left(Q_{23}+Q_{32}\right)}
\end{align*}
$$

Apparently, there are two sets of elements which can satisfy the same rotation matrix. However, in practice, the sign of $p, q$, is related to the sign of $r$, but not $s$. In the normal procedure, $s$ is computed from

$$
s=\sqrt{\left(1-p^{2}-q^{2}-r^{2}\right)}
$$

i.e., $s$ is taken as positive. In this case, there is only one set of elements which can generate the given rotation matrix.

## CRITICAL CONDITIONS FOR ORTHOGONAL MATRICES

Assuming a rotation $\theta$ (Euler angle) about an axis (Euler axis) whose direction cosines are $l, m, n$, as described in Thompson (1969), the rotation elements $a, b, c$ in matrix $A$ are related to $\theta$ and the direction cosines by the following equation:

$$
\left(\begin{array}{l}
a  \tag{28}\\
b \\
c
\end{array}\right)=2 \tan \frac{\theta}{2}\left(\begin{array}{c}
l \\
m \\
n
\end{array}\right)
$$

Therefore, if $\theta$ approaches $\pi$, then $a, b, c$ will go to infinity. In other words, the critical condition is reached when $\theta=\pi$. In the procedure of rotation matrix decomposition, this is realized as the term $\left(k_{a}^{2}+k_{b}^{2}+k_{c}^{2}\right)$ approaches zero, resulting from the large $d$ value. Note that $k_{a}=a /\left(1+a^{2}+b^{2}+c^{2}\right)$; when $a, b, c$ approach infinity, $k_{a}$ approaches zero (see Table 1).
Pope (1970) provides the relationship between the orthogonal elements of $\mathbf{Q}$ and the Euler angle and axis:

$$
\begin{align*}
p & =\cos \theta / 2 \\
q & =\sin \theta / 2 \cos \phi \cos \lambda  \tag{29}\\
r & =\sin \theta / 2 \cos \phi \sin \lambda \\
s & =\sin \theta / 2 \sin \phi
\end{align*}
$$

Table 1. The $a, b, c$ and $k_{a}, k_{b}, k_{c}$ Values

| Case | $a$ | $b$ | $c$ | $d$ | $k_{4}$ | $k_{b}$ | $k_{c}$ |
| :---: | :--- | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0.1 | 0.2 | 0.3 | 1.14 | 0.0877 | 0.1754 | 0.2631 |
| 2 | 1 | 2 | 3 | 15. | 0.0666 | 0.1333 | 0.2000 |
| 3 | $1 \times 10^{7}$ | $2 \times 10^{7}$ | $3 \times 10^{7}$ | $1.4 \times 10^{15}$ | $7.1428 \times 10^{-8}$ | $1.4285 \times 10^{-8}$ | $2.1428 \times 10^{-8}$ |

table 2. The Dual Elements and Critical Condition

| Case | Dual Elements | Critical Condition |
| :--- | :--- | :--- |
| $\mathbf{R}_{\text {ujox }}$ | $(\pi+\omega, \pi-\phi, \pi+\kappa)$ | $\cos \phi=0$ |
| $\mathbf{R}_{\text {dosk }}$ | $(\pi-\omega, \pi+\phi, \pi+\kappa)$ | $\cos \omega=0$ |
| $\mathbf{R}_{\text {ast }}$ | $(\pi+\alpha, \pi+s,-t)$ | $\sin t=0$ |
| $\mathbf{A}$ | - | $\theta=\pi$ |
| $\mathbf{Q}$ | $(-p,-q,-r,-s)$ | - |

where $\phi$ and $\lambda$ are the spherical latitude and longitude of the Euler axis. Apparently, there seems no critical condition for this formulation.

## CONCLUDING REMARKS

It has been shown clearly that generally there are two sets of rotation elements which can formulate the same rotation matrix in a formulation based on physical parameters. In other words, there are dual solutions. However, the duality does not generally hold in the formulation based on algebraic parameters.

Meanwhile, there is not always a feasible solution for the rotation matrix decomposition into the physical parameter formulation. For algebraic parameterization, the matrix formulated by a skew-symmetric matrix has the critical condition that the Euler angle should not be $\pi$. However, no critical condition is found for the orthogonal matrix formulated by unit quaternion (see Table 2).

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