Differentiator Design and Performance for Edge Sharpening

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ABSTRACT: A two-dimensional differentiator is useful for edge sharpening in digital image processing. In the design of a differentiator, differentiator coefficients that satisfy the specification of frequency response must be approximated. Four mathematical techniques—the minimax method, least-squares method, nonlinear programming, and linear programming—can be applied to solve the approximation problem. A comparison of these four differentiators was made on three types of edges: the step edge, roof edge, and line edge. The procedure for edge sharpening was first to convolve an image with the differentiator, and then to add a certain percent of the result back to the original image. Results indicated that the differentiator derived from linear programming gives the highest resolution.

INTRODUCTION

EDGE SHARPENING TECHNIQUES are used primarily as enhancement tools for highlighting object boundaries in a digital image. A differentiator is a spatial filter for determining the differentiation of an image and it can be used as a sharpening technique for edge extraction.

Theoretically, the differentiation of a given image can be determined by the Fourier transform of the image to the frequency domain, multiplying the frequency response by an operator, then transforming the result back to the spatial domain (Crowley and Parker, 1980). One problem with this approach is that adjacent images that are processed separately could exhibit discontinuities when the images are joined. In order to avoid this problem, it is useful to design a differentiator that is a set of weighting coefficients based on the specifications of the frequency responses. Edge extraction is accomplished by convolving the image with the differentiator. This approach can be applied directly to large data sets in digital image processing.

Several differentiators have been determined by the finite difference approximations of the differentiation in the spatial domain. Gonzalez and Wintz (1977) indicated that the approximation method in the spatial domain is proportional to the difference in gray levels between adjacent pixels. Thus, the gradient assumes relatively large values for prominent edges in an image and small values in image regions that are fairly smooth, being zero only in uniform regions. Crowley and Parker (1980) have studied the corresponding frequency responses of several differentiators derived from the above approach.

This paper evaluated differentiator design based on the specifications of frequency responses. Four mathematical techniques—the minimax method, least-squares method, non-linear programming, and linear programming—were applied to determine the differentiator coefficients. The common goal of these methods is to minimize an "objective" function with or without constraints. The algorithms for finding the minimal value of each objective function can be found in the referred papers. Both the minimax method and the linear programming utilize the same algorithm, but minimize different objective functions. This study compared the results of the differentiator designs that are obtained from different types of objective functions. These differentiators were then applied to three data sets of synthetic edges to compare the resolution of results. The differentiator that provided the highest resolution was applied to a SPOT image to illustrate the capability of the approach.

The objective of this paper is to present a result of edge sharpening without too much mathematical detail. However, readers can immediately use the differentiator introduced here and write code for edge sharpening. References are also provided for those who wish further details.

TWO-DIMENSIONAL DIFFERENTIATOR DESIGN

The desired frequency response, $H(u,v)$, of a differentiator is defined, without normalization, as (Gonzalez and Wintz, 1977)

$$H(u,v) = \sqrt{u^2 + v^2} \tag{1}$$

where u and v are spatial frequencies in radians/sample. In a two-dimensional discrete system, the computed frequency response, $H'(u,v)$, of an odd length differentiator with coefficients $h(m,n)$ can be written as (Fiasconaro, 1979)

$$H'(u,v) = \sum_{n=-N}^{N} \sum_{m=-N}^{N} h(m,n) e^{-j(mu+nv)} \tag{2}$$

where $j = \sqrt{-1}$ and $2N+1$ is the differentiator length in each dimension.

The approximation problem in differentiator design is to find the differentiator coefficients $h(m,n)$ that provide the best approximation $H'(u,v)$ to the desired frequency response $H(u,v)$. The edges might be anisotropic by their nature, but the isotropic form of the differentiator is assumed because all directions of edges are considered. $H(u,v)$ in Equation 1 is symmetric with respect to the line $u=v$; therefore, the study area of the approximation problem may be confined to the lower triangle, where $v \le u$ and $0 \le v \le u_N$ in which u_N is the Nyquist frequency. The differentiator coefficients $h(m,n)$ were assumed to have the same symmetric characteristics, which result in a reduction in the number of terms to be considered in the representation of $H'(u,v)$.

The error between the desired and computed frequency responses was defined as

$$E(u,v) = H(u,v) - H'(u,v). \tag{3}$$

Four mathematical techniques were used to determine differentiator coefficients by minimizing some functions related to this error. Because these four methods can also be applied to other filter designs, the term "filter" will be used in the following discussion.

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Minimax Method

The minimax method, a popular filter design method using the simplex algorithm (Luenberger, 1973), can be used to design filters by finding the maximum absolute value of the error $E(u,v)$ in the frequency domain, then minimizing it with respect to $h(m,n)$ (Rabiner *et al.*, 1975); that is, to

$$\textit{Minimize} \quad MAX \mid E(u,v) \mid . \tag{4}$$

To adequately control the smoothness of the frequency response, the desired frequency response is sampled with a very small frequency interval (Šteiglitz, 1979). In one-dimensional filter design it is possible to use the minimax method to compute coefficients easily, but this method becomes unwieldy for two-dimensional filter design as the number of frequency samples becomes large.

Least-Squares Method

The least-squares method can be applied to determine $h(m,n)$, by minimizing the L_2-norm of Equation 3; that is, to

$$\textit{Minimize} \quad \int \int E^2(u,v)dudv. \tag{5}$$

The objective function in this case is a quadratic function of the filter coefficients; hence, the minima can be found by taking the partial derivatives and setting them to zero, which leads to a system of linear equations and can be solved by the Marquardt algorithm (Marquardt, 1963). A filter designed this way is equivalent to the filter using a rectangular windowing method. However, the rectangular windowing method will lead to rippling phenomena if the desired frequency response $H(u,v)$ has discontinuities (Oppenheim and Schafer, 1975).

Nonlinear Programming

Gold and Jordan (1969) introduced a direct search method to the approximation problem. In their approach, the frequency response is specified exactly at M equidistant frequencies in each dimension. If the number of frequency samples is equal to the number of weighting coefficients, then the weighting coefficients can be exactly determined. However, the frequency responses of any two adjacent specified samples cannot be controlled independently and may not behave as intended.

A previous study of filter design (Lewin and Telljohann, 1984) indicated that the smoothness of the frequency response may be controlled by forcing the magnitudes of the coefficients to decrease gradually from the center of the filter to the edges. Pan (1988) proposed a nonlinear programming technique that forms a compromise between matching the desired frequency response with the computed frequency response, and by smoothing the variation between two adjacent samples. In that approach, an exponential weighting factor is used to control the magnitudes of the filter coefficients. The filter coefficients can be determined by minimizing a nonlinear function of $h(m,n)$; that is, to

$$\textit{Minimize} \quad pP(\mathbf{X}) + qQ(\mathbf{X}) \tag{6}$$

where $\mathbf{X}$ is a vector with components $h(\mathrm{m,n})$, and

$$P(\mathbf{X}) = \int \int E^2(u,v)dudv$$

$$Q(\mathbf{X}) = -h^2(0,0) + \sum_{n=-N}^{N} \sum_{\substack{n=m \\ n \neq 0}}^{N} h^2(m,n)e^{z\sqrt{m2+n2}}$$

in which p, q, and z are positive constants.

The first term, $P(\mathbf{X})$, in the above objective function, is a least-squares term that forces the solution into satisfying the desired frequency response. This is done by treating the equation as a penalty constraint weighted by a factor p. The second term, $Q(\mathbf{X})$, incorporates the spatial constraints weighted by factor q. The larger the ratio p/q, the greater the emphasis on exerting the accuracy of frequency response relative to the magnitude of variation of the filter. If q is set to zero, then the objective function is identical to that used in the least-squares method.

The exponential-type function in the second term weights filter coefficients more when their distance is further from the central coefficient. The central coefficient, $h(0,0)$, is not weighted. The weight z in the exponential factor is selected on the basis of the characteristics of the filter. Powell's (1964) nonlinear programming algorithm is used to find the desired vector $\mathbf{X}$.

Linear Programming

A technique modified from Lewin and Telljohann (1984) can be applied to minimize the error $E(u,v)$ while controlling the smoothness of frequency response. The principal advantage of using linear programming to solve the problem is that the maximum error of the computed frequency response and the magnitudes of the filter coefficients can be controlled simultaneously.

The filter coefficients are determined by minimizing a function of $|h(m,n)|$; that is, to

$$\textit{Minimize}\ R(\mathbf{X})$$

where

$$R(\mathbf{X}) = - \mid h(0,0) \mid + \sum_{n=-N}^{N} \sum_{\substack{n=m \\ n \neq 0}}^{N} \mid h(m,n) | e^{r\sqrt{m2+n2}}$$

X is a vector of $h(m,n)$ and r is a positive constant. Before using the simplex method to minimize $R(X)$, each $\mid h(m,n) \mid$ has to be represented by two non-negative variables (Lewin and Telljohann, 1984), so that the objective function can be modified to be a linear function of these variables. This minimization is subject to the constraint, $\mid E(u,v) \mid \leq W(u,v)$, where the error tolerance $W(u,v)$ is taken to be very small relative to $H(u,v)$.

The exponential factor in Equation 7 has the same effect as that used in Equation 6. The factor r in the exponential function is user selected and can be adjusted to force the magnitudes of filter coefficients at the edges to very small numbers.

Results of Differentiator Design

Theoretically, the least-squares method and nonlinear programming are used to solve the overdetermined problem, in which the number of sampled data is greater than the number of unknowns. In contrast, the minimax method and linear programming are used to solve the underdetermined problem. Frequency samples are specified at M equidistant frequencies along each dimension in the first quadrant of the uv-plane. A value of 5 for the differentiator size N is used in these four methods, but the sample size M is 6 for the least-squares method and nonlinear programming, and M is 4 for the minimax method and linear programming.

Figure 1 shows the desired frequency responses of a differentiator (Equation 1), and Table 1 lists the differentiator coefficients determined from the four methods. The sum of coefficients is the total of all coefficients, and should be zero for an ideal differentiator. The error at each sampled frequency was less than 0.01 percent of $H(u,v)$.

Because no discontinuity occurred in the desired frequency responses, the differentiator derived from the least-squares method was applicable to edge sharpening. However, the least-squares method may not be appropriate for filter design where discontinuity occurs. The parameters p, q, and z used in nonlinear programming (Equation 6) are 1,000, 1, and 1, respectively. As shown in Table 1, the differentiator designed from

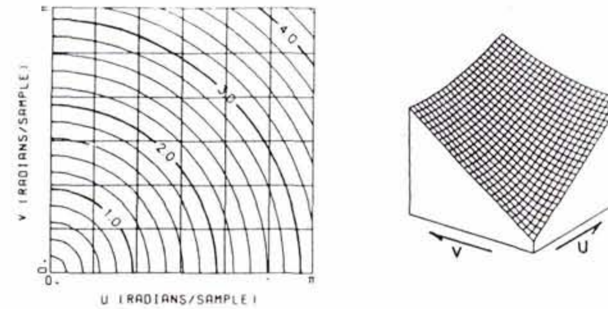

FIG. 1. A contour map and the corresponding perspective view of the frequency responses of an ideal differentiator.

nonlinear programming is quite close to that of the least-squares method.

Figure 2 shows the computed frequency responses from each differentiator. As shown in Figure 2a, the result from the minimax method has significant "ripple" features occurring between two specified samples. By minimizing the objective function as given in Equation 7, the ripple phenomenon can be reduced, where the parameter r was set to 1.

The theory of linear programming states that the minimal value of an objective function can be determined by using only the same number of variables that are required to satisfy the constraints; the remaining variables were set to zero (Luenberger, 1973). More independent differentiator coefficients exist than were specified samples in this design problem; hence, six of the coefficients in the derived differentiator were zero.

However, by minimizing $R(X)$ as given in Equation 7, the actual size of the differentiator was reduced by 1. This cannot be achieved by using the minimax method or the approach adopted by Lewin and Telljohann (1984). The sum of coefficients was 0.00003, where the theoretical value should be zero, which made the result more accurate than the other three differentiators. Also, the smaller differentiator size made the convolution faster.

The computational time for each method was dependent upon the filter size (N) and sample size (M). Because both the minimax method and linear programming use the simplex algorithm to solve the approximation problem, the computation times of these two methods were almost identical, given the same filter size and sample data. In the example in Table 1, the computation time was 5 seconds using either of these two methods. Either the least-squares or nonlinear programming method used 34 seconds, because the required set was larger than that used in the minimax method and linear programming.

APPLICATIONS

The part of a digital image with similar gray levels can be thought of as a region, and an edge is the intersection of two adjacent regions. The edge separates regions with different features and is therefore useful for the identification of objects in images.

Generally there are three types of edges: step, roof, and line edges (Nalwa and Binford, 1986). The step edge occurs when the gray levels of two adjacent regions vary abruptly at the intersection. The roof edge occurs when a gradual change in gray level reaches a point of inflection and then decreases in gray level. The line edge occurs when a narrow band of a dif-

TABLE 1. THE DIFFERENTIATOR COEFFICIENTS DETERMINED FROM FOUR MATHEMATICAL METHODS

(1) Minimax method (sum of coefficients = 0.00006)						
n = 5	−0.06019	0.00000	0.00000	−0.00308	0.00000	0.00000
4	0.01800	−0.00182	−0.00329	−0.00253	−0.00135	0.00000
3	0.00000	−0.01237	−0.00635	0.00000	−0.00253	−0.00308
2	0.06410	−0.01114	−0.01127	−0.00635	−0.00329	0.00000
1	−0.45148	−0.08412	−0.01114	−0.01237	−0.00182	0.00000
0	2.42994	−0.45148	0.06410	0.00000	0.01800	−0.06019
m =	0	1	2	3	4	5
(2) Least-square method (sum of coefficients = 0.00081)						
n = 5	−0.02309	−0.00244	−0.00211	−0.00270	−0.00192	−0.00216
4	0.02298	−0.00144	−0.00205	−0.00233	−0.00300	−0.00192
3	−0.04573	−0.00948	−0.00407	−0.00162	−0.00233	−0.00270
2	0.05766	−0.00990	−0.00982	−0.00407	−0.00205	−0.00211
1	−0.44191	−0.08149	−0.00990	−0.00948	−0.00144	−0.00244
0	2.41943	−0.44191	0.05766	−0.04573	0.02298	−0.02309
m =	0	1	2	3	4	5
(3) Nonlinear programming (sum of coefficients = 0.00023)						
n = 5	−0.02303	−0.00252	−0.00213	−0.00281	−0.00185	−0.00156
4	0.02303	−0.00147	−0.00206	−0.00234	−0.00301	−0.00185
3	−0.04572	−0.00944	−0.00408	−0.00165	−0.00234	−0.00281
2	0.05763	−0.00989	−0.00984	−0.00408	−0.00206	−0.00213
1	−0.44190	−0.08153	−0.00989	−0.00944	−0.00147	−0.00252
0	2.41927	−0.44190	0.05763	−0.04572	0.02303	−0.02303
m =	0	1	2	3	4	5
(4) Linear programming (sum of coefficients = 0.00003)						
n = 5	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
4	0.01800	−0.00182	−0.00329	−0.00253	−0.00135	0.00000
3	−0.06020	−0.01237	−0.00635	−0.00616	−0.00253	0.00000
2	0.06410	−0.01114	−0.01127	−0.00635	−0.00329	0.00000
1	−0.45148	−0.08412	−0.01114	−0.01237	−0.00182	0.00000
0	2.42995	−0.45148	0.06410	−0.06020	0.01800	0.00000
m =	0	1	2	3	4	5

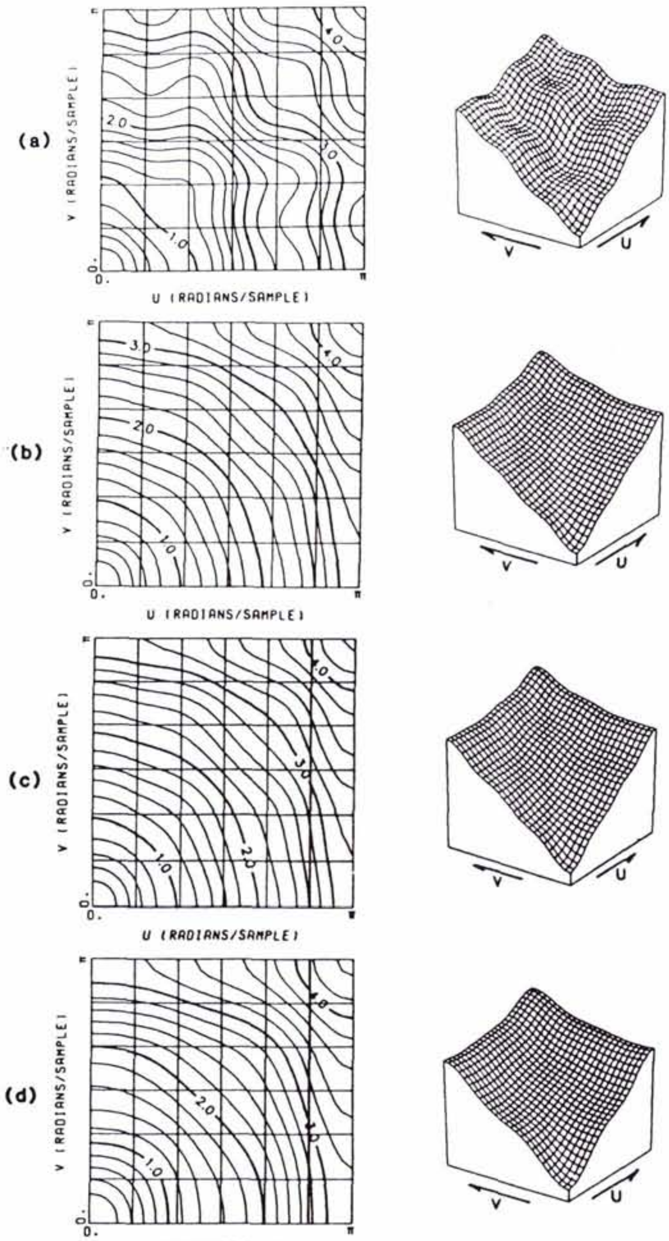

FIG. 2. Contour maps and perspective views of the differentiators derived from the following methods: (a) Minimax method; (b) Least-squares method; (c) Nonlinear programming; and (d) Linear programming.

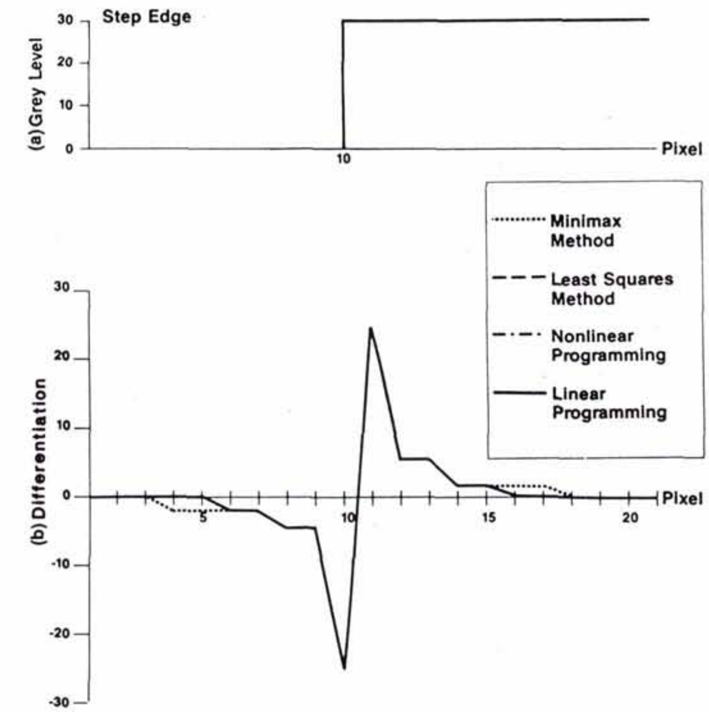

FIG. 3. The four differentiators applied to a step edge. The results determined from least-squares method, nonlinear programming, and linear programming were almost the same.

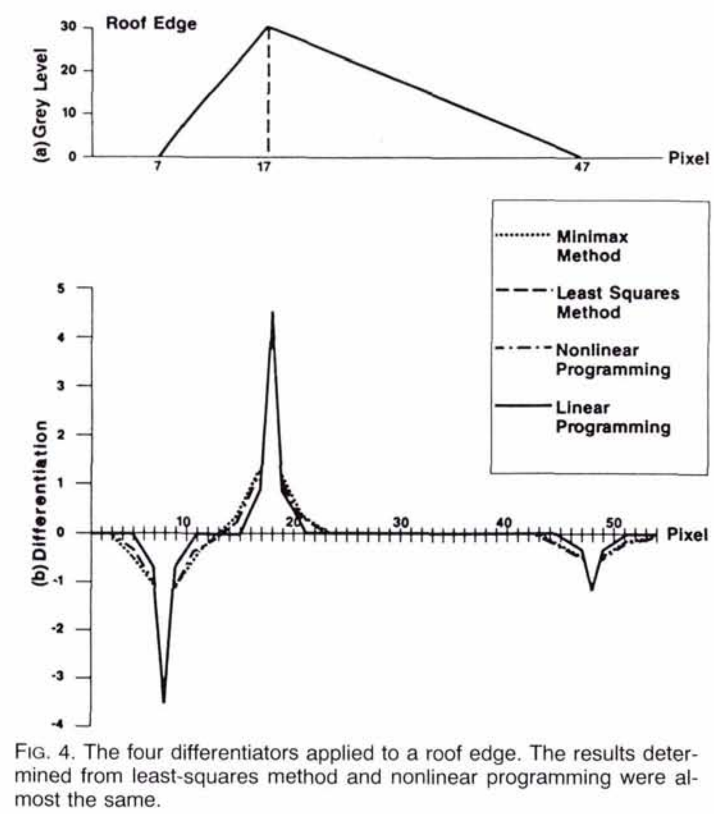

FIG. 4. The four differentiators applied to a roof edge. The results determined from least-squares method and nonlinear programming were almost the same.

ferent gray level occurs within or between regions of another gray level.

The four differentiators designed were implemented on these three synthetic edges. Figures 3, 4, and 5 show the profiles of step edge, roof edge, and line edge, respectively, and the results of convolution with each differentiator. In Figure 3, all differentiators have almost the same resolutions, except the differentiator derived from the minimax method has a wider range. In Figure 4, the differentiator derived from the linear programming has a finer resolution. The line edge shown in Figure 5 has a width of 5 pixels. In fact, a line edge can be obtained by the subtraction of two step edges. The result was similar to the performance of the step edge.

Figures 3, 4, and 5 show that the differentiator derived from linear programming gave higher resolution when applied to these three types of edges. This differentiator was demonstrated using a 800 by 800-pixel SPOT panchromatic image of a section of the Green River near Vernal, Utah (Figure 6a). The

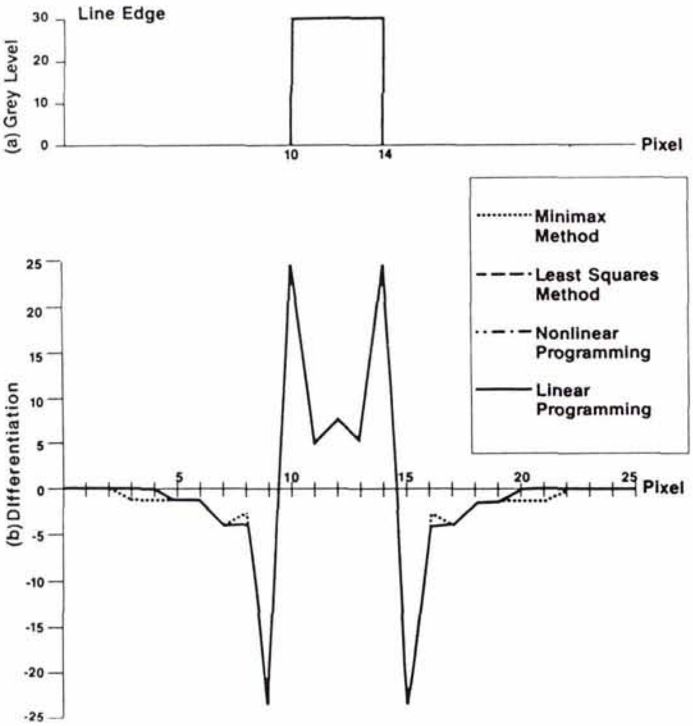

FIG. 5. The four differentiators applied to a line edge. The results determined from least-squares method, nonlinear programming, and linear programming were almost the same.

image resulting from the convolution with the differentiator has gray levels with both positive and negative values. Figure 6b shows this image after a shift was applied to it to make all values positive. A percentage of this shifted image was then added back to the original image to enhance it. Figure 6c is the result of a 30 percent addback, and Figure 6d resulted from a 60-percent addback. The ideal percentage was subjective and varies from image to image.

The sharpened image can be justified by selecting either a horizontal or vertical profile (not shown here) and comparing it with that from the original image. The difference of gray level at each edge will be increased after sharpening. Because the differentiator is a type of high-pass filter, many fine edged features can be identified. However, when an image is contaminated with noise, this differentiator is not applicable for edge sharpening directly, because noise generally involves high frequency.

CONCLUSIONS

Four mathematical techniques have been applied to design a differentiator based on the specifications of frequency response. Both linear and nonlinear programming techniques provide the advantage of simultaneously satisfying the constraints of both the spatial and frequency domains. The differentiator derived from minimizing the objective function in Equation 7 by linear programming gave the highest resolution in the synthetic edges tests. The edge sharpening was accomplished by convolving the differentiator with a given image and adding a percent of the result back to the original image. The techniques discussed in this paper can be applied to various filter designs that are based on the specifications of desired frequency responses.

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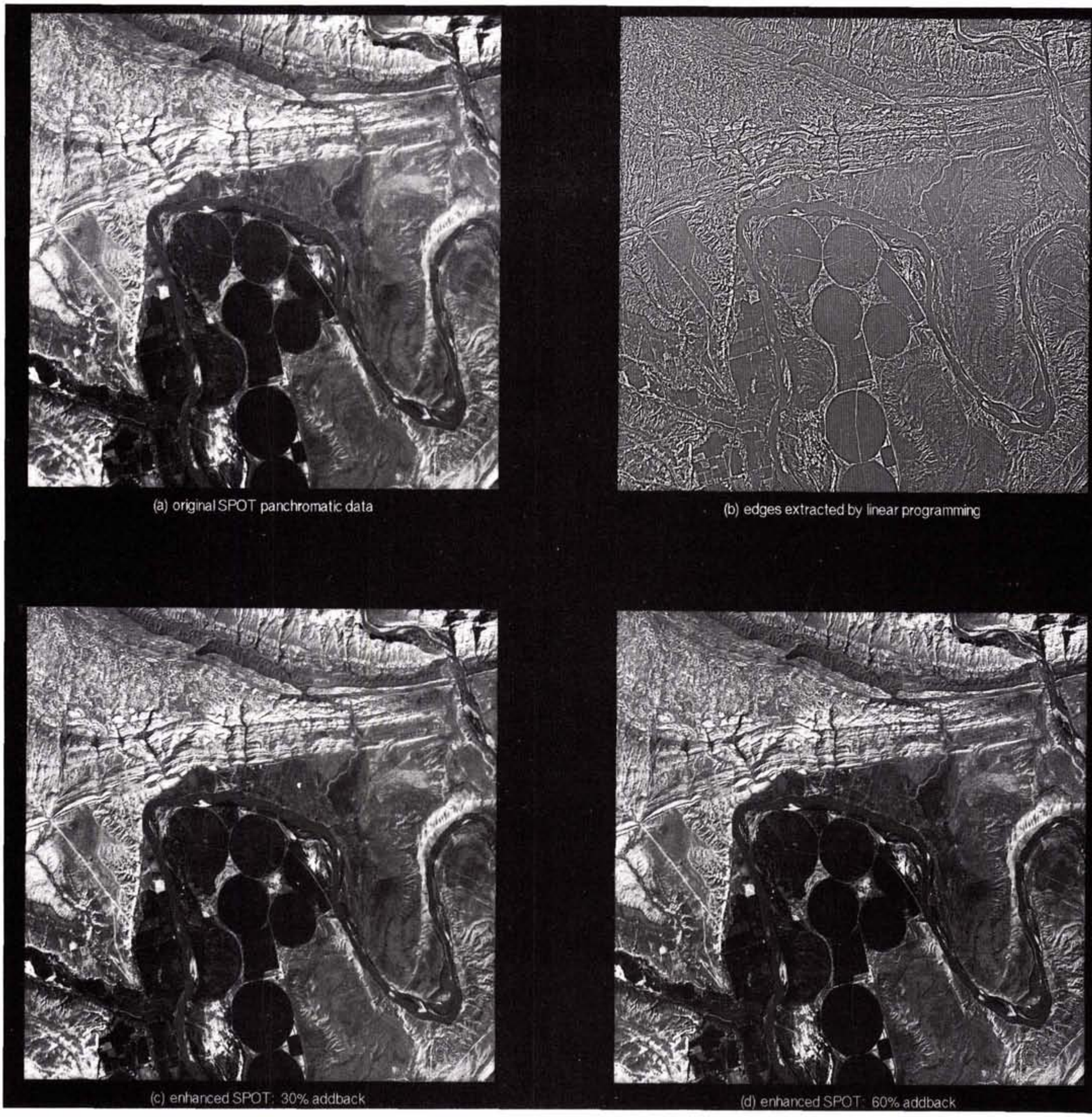

FIG. 6. The performance of a differentiator derived from linear programming on a SPOT panchromatic image. (a) Original SPOT image of a section of the Green River near Vernal, Utah (Copyright SPOT data CNES). (b) Edge extraction from (a). (c) Add 30 percent of (b) back to (a). (d) Add 60 percent of (b) back to (a).