On the Duality of Relative Orientation

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ABSTRACT: For a stereo-pair, there are two sets of relative orientation parameters which satisfy the coplanarity condition with five specified parameters, e.g., $\mathbf{B}_Y$, $\mathbf{B}_Z$, $\Delta\omega$, $\Delta\phi$, $\Delta\kappa$ at the same time. This phenomenon results from the duality of coplanarity condition. With the aid of RLT, an algebraic model of relative orientation, the characteristics of the duality in the coplanarity condition and relative orientation are investigated.

INTRODUCTION

RELATIVE ORIENTATION is conventionally realized by the coplanarity equations, which are non-linear with respect to the unknown parameters. Therefore, an iterative approach is required for the solution, and initial values are needed. Due to the specification of initial values, the direction of convergence and even the region of the final estimates are pre-defined before the iteration process starts. Therefore, the duality of the coplanarity condition has been scarcely experienced in practice. With the aid of RLT (Relative orientation with Linear Transform), a closed-form solution of relative orientation through an algebraic mathematical model, this phenomenon can be explored in detail.

DUALITY AND THE RDLT MODEL

Conventionally, relative orientation is analytically realized by the coplanarity condition: the base vector ($\mathbf{B}_X$, $\mathbf{B}_Y$, $\mathbf{B}_Z$), and the two corresponding image vectors (**U, V, W**) and (**U′, V′, W′**) for any point pair lie in the same plane. Mathematically, this can be presented as follows:

$$\begin{vmatrix} \mathbf{B}_X & \mathbf{B}_Y & \mathbf{B}_Z \\ \mathbf{U} & \mathbf{V} & \mathbf{W} \\ \mathbf{U}' & \mathbf{V}' & \mathbf{W}' \end{vmatrix} = 0$$

where

$$\begin{vmatrix} \mathbf{U} \\ \mathbf{V} \\ \mathbf{W} \end{vmatrix} = \begin{vmatrix} x_i - x^o + dx \\ y_i - y_o + dy \\ -f \end{vmatrix} = \begin{vmatrix} x \\ y \\ -f \end{vmatrix} \tag{1}$$

$$\begin{vmatrix} \mathbf{U}' \\ \mathbf{V}' \\ \mathbf{W}' \end{vmatrix} = \begin{vmatrix} x_i - x^{o\prime} + dx' \\ y_i' - y_o' + dy' \\ -f' \end{vmatrix} = \begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} \begin{vmatrix} x' \\ y' \\ -f' \end{vmatrix} = \mathbf{R} \begin{vmatrix} x' \\ y' \\ -f' \end{vmatrix}, \text{ and}$$

R is an orthogonal matrix.

Relative orientation can also be realized with the collinearity equations. That is, the vectors formed by two image points of a point pair and their corresponding perspective centers intersect at the same point: the object/model point.

The third mathematical model for relative orientation is the RLT (Relative orientation with Linear Transform) model (Khlebnikova, 1983; Chang, 1986). RLT can be derived by expanding Equation 1: i.e.,

$$L_1 y\, x' + L_2 y\, y' - L_3 y\, f' + L_4 f\, x' + L_5 f\, y' - L_6 f\, f' + L_7 x\, x' + L_8 x\, y' - L_9 x\, f' = 0. \tag{2}$$

Eliminating L_5 from Equation 2, we obtain:

$$L_1' y\, x' + L_2' y\, y' - L_3' y\, f' + L_4' f\, x' - L_6' f\, f' + L_7' x\, x' + L_8' x\, y' - L_9' x\, f' + f\, y' = 0 \tag{3}$$

where $L_i' = L_i/L_5$.

This is the basic model of RLT. When the misclosure of this equation is subjected to optimization, this model is linear. A direct solution for these eight parameters can be achieved without knowing any approximate values.

With given B_X, L_5 and the base components can be obtained as

$$L_5^2 = 2B_X^2/((L_1'^2 + L_2'^2 + L_3'^2) + (L_4'^2 + L_5'^2 + L_6'^2) - (L_7'^2 + L_8'^2 + L_9'^2));$$

$$B_Y = -\frac{L_1 L_7 + L_2 L_8 + L_3 L_9}{B_X} \qquad B_Z = \frac{L_4 L_7 + L_5 L_8 + L_6 L_9}{B_X} \tag{4}$$

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The sign problem is introduced here, because L_5 may have either positive or negative values. This brings us the duality of relative orientation parameters, which is presented in Equation 5, where the sign of L_5 affects the rotation elements.

$$r_{11} = \frac{L_3L_5 - L_2L_6 - B_ZL_1 - B_YL_4}{B_X^2 + B_Y^2 + B_Z^2}; \quad r_{21} = \frac{B_Y r_{11} + L_4}{B_X}; \quad r_{31} = \frac{B_Z r_{11} + L_1}{B_X};$$
$$r_{12} = \frac{L_1L_6 - L_3L_4 - B_ZL_2 - B_YL_5}{B_X^2 + B_Y^2 + B_Z^2}; \quad r_{22} = \frac{B_Y r_{12} + L_5}{B_X}; \quad r_{32} = \frac{B_Z r_{12} + L_2}{B_X}; \tag{5}$$
$$r_{13} = \frac{L_2L_4 - L_1L_5 - B_ZL_3 - B_YL_6}{B_X^2 + B_Y^2 + B_Z^2}. \quad r_{23} = \frac{B_Y r_{13} + L_6}{B_X}; \quad r_{33} = \frac{B_Z r_{13} + L_3}{B_X}.$$

DISCOVERING THE DUAL SOLUTION

From the assignment of the sign of L_5, the dual solutions for six simulated examples are tested. Five parameters: B_Y, B_Z, $\Delta\omega$, $\Delta\phi$, $\Delta\kappa$ are used to characterize the test samples, and throughout all tests, B_X is specified as unity. The resulting base components are identical in both negative and positive L_5 cases, as expected. The resulting rotation matrix and decomposed elements are listed in the following, where $\mathbf{R}_i^p$ denotes the rotation matrix of **Test i**, with positive L_5, $\mathbf{R}_i^n$ denotes the one with negative L_5.

Test 1:

$(B_Y, B_Z, \Delta\omega, \Delta\phi, \Delta\kappa) = (0., 0., 0., 0., 0.);$

Positive L_5:

$$\mathbf{R}_1^p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$(\omega, \phi, \kappa) = (0., 0., 0.).$

Negative L_5:

$$\mathbf{R}_1^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

$(3.14159, 0., 0.).$

Test 2:

$(B_Y, B_Z, \Delta\omega, \Delta\phi, \Delta\kappa) = (0.1, 0.2, 0., 0., 0.);$

Positive L_5:

$$\mathbf{R}_2^p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$(\omega, \phi, \kappa) = (0., 0., 0.).$

Negative L_5:

$$\mathbf{R}_2^n = \begin{pmatrix} 0.90476 & 0.19047 & 0.38095 \\ 0.19047 & -0.98095 & 0.03809 \\ 0.38095 & 0.03809 & -0.92380 \end{pmatrix};$$

$(-3.10038, 0.39083, -0.20750).$

Test 3:

$(B_Y, B_Z, \Delta\omega, \Delta\phi, \Delta\kappa) = (0.1, 0.2, 0.3, 0.4, 0.5);$

Positive L_5:

$$\mathbf{R}_3^p = \begin{pmatrix} 0.80831 & -0.44158 & 0.38942 \\ 0.55900 & 0.78321 & -0.27219 \\ -0.18480 & 0.43770 & 0.87992 \end{pmatrix};$$

$(\omega, \phi, \kappa) = (0.3, 0.4, 0.5).$

Negative L_5:

$$\mathbf{R}_3^n = \begin{pmatrix} 0.76740 & -0.08360 & 0.63569 \\ -0.40143 & -0.83573 & 0.37470 \\ 0.49994 & -0.54274 & -0.67490 \end{pmatrix};$$

$(-2.63477, 0.68891, 0.10851).$

Test 4:

$(B_Y, B_Z, \Delta\omega, \Delta\phi, \Delta\kappa) = (0.1, 0.2, 1., 2., 3.);$

Positive L_5:

$$\mathbf{R}_4^p = \begin{pmatrix} 0.47351 & -0.36026 & 0.80374 \\ 0.76979 & 0.61272 & -0.17887 \\ -0.42803 & 0.70341 & 0.56745 \end{pmatrix};$$

$(\omega, \phi, \kappa) = (0.30536, 0.93356, 0.65039).$

Negative L_5:

$$\mathbf{R}_4^n = \begin{pmatrix} 0.41198 & 0.05873 & 0.90930 \\ -0.68124 & -0.64287 & 0.35018 \\ 0.60513 & -0.76372 & -0.22484 \end{pmatrix};$$

$(-2.14159, 1.14159, -0.14159).$

Test 5:

$(B_Y, B_Z, \Delta\omega, \Delta\phi, \Delta\kappa) = (-0.1, -0.2, 0., 0., 0.);$

Positive L_5:

$$\mathbf{R}_5^p = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$(\omega, \phi, \kappa) = (0., 0., 0.).$

Negative L_5:

$$\mathbf{R}_5^n = \begin{pmatrix} 0.90476 & -0.19047 & -0.38095 \\ -0.19047 & -0.98095 & 0.03809 \\ -0.38095 & 0.03809 & -0.92380 \end{pmatrix};$$

$(-3.10038, -0.39083, 0.20750).$

Test 6:

$(B_Y, B_Z, \Delta\omega, \Delta\phi, \Delta\kappa) = (-0.1, -0.2, 1., 2., 3.);$

Positive L_5:

$$\mathbf{R}_6^p = \begin{pmatrix} 0.27198 & 0.46652 & 0.84165 \\ 0.61285 & 0.59035 & -0.52527 \\ -0.74192 & 0.65866 & 0.12534 \end{pmatrix};$$

$(\omega, \phi, \kappa) = (1.80504, 1.00034, -1.04298).$

Negative L_5:

$$\mathbf{R}_6^n = \begin{pmatrix} 0.41198 & 0.05873 & 0.90930 \\ -0.68124 & -0.64287 & 0.35018 \\ 0.60513 & -0.76372 & -0.22484 \end{pmatrix};$$

$(-2.14159, 1.14159, -0.14159).$

All these dual solutions can be verified using a relative orientation procedure based on the coplanarity equation model.

PHYSICAL INTERPRETATION

As demonstrated from the previous numerical tests, there are dual solutions for the coplanarity condition with specified five parameters. However, only one of them represents the proper model space. This can be illustrated with Figure 1.

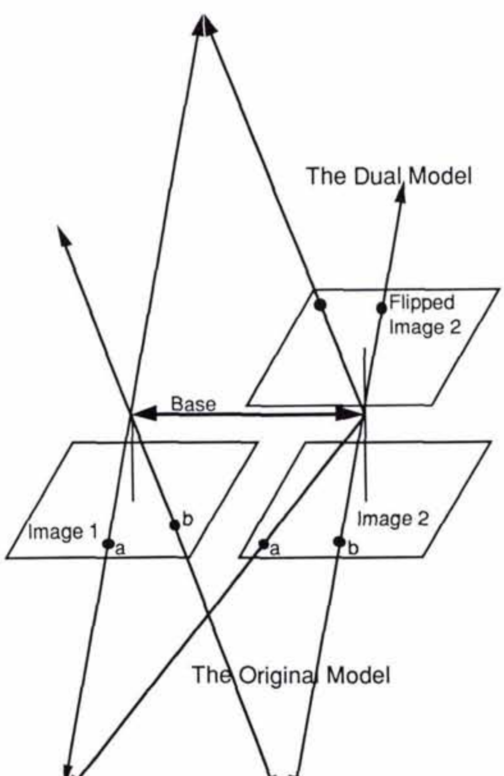

FIG. 1. The dual solution of the coplanarity condition.

From Figure 1, the cause of the dual solution can be visualized. Test 1 also provides a straightforward illustration. In the $(B_Y, B_Z, \Delta\omega, \Delta\phi, \Delta\kappa) = (0., 0., 0., 0., 0.)$ case, the dual solution is simply to flip the second photo over around the X axis, the direction of the second bundle is then changed. Correspondingly, another intersection is defined.

Actually, because the two intersections and the perspective center of the second image define a unique plane, all these three vectors are coplanar. While the base components are not equal to zero, the rotation matrix models the influence from base components as well. This makes the numerical value of angles relatively difficult to visualize.

Analysis of the test examples shows that, in Tests 1, 2, 3, and 5, the positive L_5 recovers the given relative orientation; while in Tests 4 and 6, the negative L_5 recovers the given parameters. One may be surprised that in Tests 4 and 6, $(\omega - \pi, \pi - \phi, \kappa - \pi)$ is obtained, rather than (ω, ϕ, κ). However, these two sets formulate the identical rotation matrix in the $\mathbf{R}_{\omega\phi\kappa}$ sequence, and this phenomenon is known as the duality of the rotation matrix (Shih, 1990).

Comparing Tests 1 and 2 shows that the rotation between the dual solutions is not a simple rotation around the X axis. The difference between $\mathbf{R}^n{}_1$ and $\mathbf{R}^n{}_2$ results from the base components. Further comparing $\mathbf{R}^n{}_2$ and $\mathbf{R}^n{}_3$, it was found that

$$\mathbf{R}_4^n = \mathbf{R}_2^n \mathbf{R}_3^p.$$

The physical explanation of this phenomenon is that the dual solution is rotating around the base vector, rather than the X axis, by π. This can also be seen from

$$\mathbf{R}_4^p = \mathbf{R}_2^n \mathbf{R}_4^n$$

$$\mathbf{R}_6^n = \mathbf{R}_5^n \mathbf{R}_6^p.$$

Comparing Tests 2 and 5, one can discover that a sign change of the base components results in a sign change of ϕ and κ in the rotation matrix of the dual solution. It should be noted that the decomposition of rotation elements is performed according to $\mathbf{R}_{\omega\phi\kappa}$.

CONCLUDING REMARKS

The existence of a dual solution in the coplanarity condition has been explored and investigated. The difference between the two solutions is the resulting rotation matrix. Physically, the duality of the coplanarity condition is understood as a rotation of the second image of the stereo-pair around the base vector by π providing another valid solution. However, similarity relations do not hold in the model spaces derived from these two solutions.

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